Foundations for type-driven probabilistic modelling

Ohad Kammar
University of Edinburgh

Logic Summer School
Australian National University
4–16 December, 2023
Canberra, ACT, Australia

supported by: The Royal Society, The Alan Turing Institute, Facebook Research, NCSC

Plan:

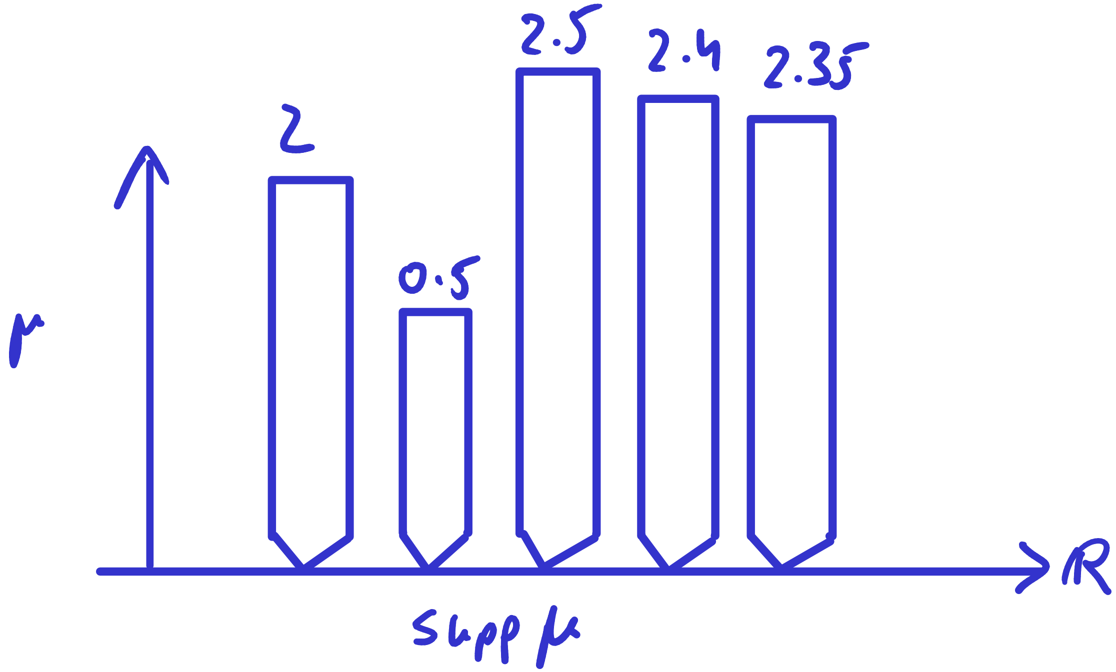
1) type-driven probability: discrete case (Mon + Tue)
2) Borel sets & measurable spaces (Wed)
3) Quasi Borel spaces (Wed) Simple type structure (Thu)
4) Dependent type structure & standard Borel spaces (Thu)
5) Integration & random variables (Fri)

Please ask questions!

Course web page

Scribble

discrete model measure only histograms:



Want:

- lengths
- areas
- volumes.

Continuous Caveat:

Thm: no $\lambda : \mathcal{P}\mathbb{R} \to [0,\infty]$:

$$\lambda(a,b) = b-a \qquad \lambda(r+A) = \lambda A \qquad \lambda\left(\biguplus_{n=0}^{\infty} A_n\right) = \sum_{n=0}^{\infty} \lambda A_n$$

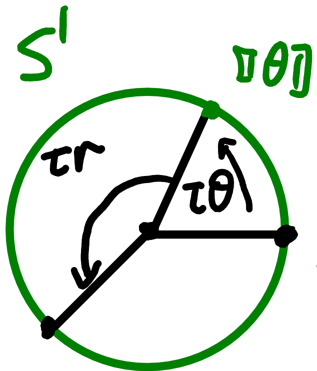
(generalises length) (translation invariant) σ-additive

Thm: no $\lambda : \mathcal{P}\mathbb{R} \to [0,\infty]$:

$$\lambda(a,b) = b - a \qquad \lambda(r + A) = \lambda A \qquad \lambda\left(\biguplus_{n=0}^{\infty} A_n\right) = \sum_{n=0}^{\infty} \lambda A_n$$

(generalises length) (translation invariant) σ-additive

Direct proof in standard analysis courses. Idea behind typical proof is:



$[\![\theta]\!] := (\cos\theta, \sin\theta)$

$[\![0]\!] := (1,0)$

Thm: no $\lambda : \mathcal{P}S^1 \to [0,\infty]$ s.t.

$r : \mathbb{R} \vdash \text{rotate}_r [\![\theta]\!] := [\![\theta + \tau r]\!]$

a) Satisfy measure axioms for $\mathcal{B}S^1 := \mathcal{P}S^1$

b) invariant under rotations: $E : \mathcal{B}S^1 \vdash$

$$\lambda\, \text{rotate}[E] = \lambda E$$

c) $\lambda S^1 = \tau \; (:= 2\pi)$

reduce (S^1, λ^{S^1}) to $(\mathbb{R}, \lambda^{\mathbb{R}})$ via restriction & push forward

$$\lambda^{\mathbb{R}}\big|_{\mathcal{P}[0,1)} := \lambda E \subseteq [0,1].\ \lambda E : \mathcal{P}[0,1) \to \mathbb{W}$$

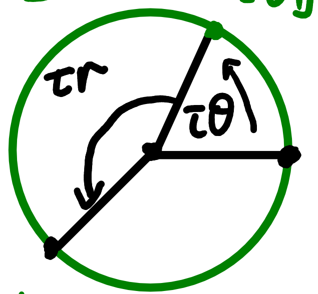
$$\lambda^{S^1} := \lambda E \subseteq S^1.\ \lambda^{\mathbb{R}}_{\mathcal{P}[0,1)}\left(⟦-⟧^{-1}[E]\right) : \mathcal{P}S^1 \xrightarrow{⟦-⟧^{-1}} \mathcal{P}[0,1) \xrightarrow{\lambda^{\mathbb{R}}_{[0,1)}} \mathbb{W}$$

noting

rotations in $S^1 \longleftrightarrow$ translations in $\mathbb{R}$

Since $\nexists\, \lambda^{S^1}$, we have $\nexists\, \lambda^{\mathbb{R}}$ either.

S^1 $\quad [\![\theta]\!] := (\cos\theta, \sin\theta)$



$[\![0]\!] := (1,0)$

$r : \mathbb{R} \vdash \mathrm{rotate}_r [\![\theta]\!] := [\![\theta + \tau r]\!]$

Thm: no $\lambda : \mathcal{P}S^1 \to [0, \infty]$

s.t.

a) satisfy measure axioms for $\mathcal{B}S^1 := \mathcal{P}S^1$

b) invariant under rotations: $E : \mathcal{B}S^1 \vdash \lambda\, \mathrm{rotate}[E] = \lambda E$

c) $\lambda S^1 = \tau \; (:= 2\pi)$

Proof: $a + b \Rightarrow \neg c$:

1) Using axiom of choice (AoC):

$$S^1 = \biguplus_{i=0}^{\infty} E_i \qquad E_i = \mathrm{rotate}_{r_i}[E_0]$$

2) $$\lambda S^1 = \sum_{i=0}^{\infty} \lambda E_i = \sum_i \lambda\, \mathrm{rotate}_{r_i} E_0 = \sum_{i=0}^{\infty} \lambda E_0 = \begin{cases} \lambda E_0 = 0 : 0 \\ \lambda E_0 > 0 : \infty \end{cases} \neq \tau$$

Constructing E_i:

$$x, y : S' \vdash x \sim y := \exists q \in \mathbb{Q}.\ \mathrm{rotate}_q\, x = y \qquad : \mathrm{Prop}$$

$$\equiv \exists q \in [0,1) \cap \mathbb{Q}.\ \mathrm{rotate}\, x = y$$

$\sim$-equivalence classes:

$$x : S' \vdash [x]_\sim := \{ y \in S' \mid x \sim y \} \qquad : \mathcal{P} S'$$

$$C := \{ [x]_\sim \in \mathcal{P} S' \mid x \in S' \}$$

$\forall e \in C,\ e \neq \emptyset$, so by AoC: $\exists \xi_- : C \to S'.\ \xi_e \in e.$

NB: ξ injective

Take $C_0 := \{\xi_e \in S' \mid e \in C\} \in \mathcal{P} S'$

Note: $x \sim y,\ x, y \in C_0 \vdash x = y$.

$q : \mathbb{Q} \vdash C_q := \mathrm{rotate}_q[C_0] \in \mathcal{P} S'$

NB: $x, y : S' \vdash$

$x \sim y : \mathrm{Prop}$

C = $\sim$-equiv.

$\xi_- : C \rightarrowtail S'$

$e : C \vdash \xi_e \in e$

Let $(r_i)_{i=0}^{\infty}$ enumerate $\mathbb{Q} \cap [0, 1)$ s.t. $r_0 = 0$

Take $E_i := C_{r_i}$

By fiat: $E_i = C_{r_i} = \mathrm{rotate}_{r_i}[C_0] = \mathrm{rotate}_{r_i}[E_0]$

RTP: $S' = \bigoplus_{i=0}^{\infty} E_i$

$E_i \cap E_j = \emptyset, \quad i \neq j:$

$x \in E_1 \cap E_2 \Rightarrow \exists\, y_i \in C_0 .\ x = \mathrm{rotate}_{r_i} y_i$

$\Rightarrow y_1 \sim x \sim y_2 \Rightarrow y_1 = y_2 =: y$

$\Rightarrow \mathrm{rotate}_{r_2 - r_1} y = y, \ |r_2 - r_1| < 1$

$\Rightarrow r_1 = r_2$

$\underline{S'} = \bigcup_{i=0}^{\infty} \underline{E_i}$: $x \in S'$. letting $e := \xi_{[x]_\sim} : PS'$

$\xi_e, x \in e \Rightarrow \xi_e \sim x$

$\Rightarrow \exists q \in \mathbb{Q} \cap [0,1).\ \mathrm{rotate}_q\, \xi_e = x.$

As $\xi_e \in C_0$: $x \in C_q$. Find i s.t. $r_i = q$

and $x \in C_{r_i} = E_i$. $\blacksquare$

Takeaway: taking $\mathcal{B}\mathbb{R} := \mathcal{P}\mathbb{R}$

excludes measures such as:

length, area, volume

Workaround: only measure well-behaved subsets

Def: The Borel subsets $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{P}\mathbb{R}$:

- open intervals $(a,b) \in \mathcal{B}_{\mathbb{R}}$

Closure under σ-algebra operations:

$$\frac{}{\emptyset \in \mathcal{B}_{\mathbb{R}}} \qquad \frac{A \in \mathcal{B}_{\mathbb{R}}}{A^c := \mathbb{R} \setminus A \in \mathcal{B}} \qquad \frac{\vec{A} \in \mathcal{B}_{\mathbb{R}}^{\mathbb{N}}}{\bigcup_{n=0}^{\infty} A_n \in \mathcal{B}_{\mathbb{R}}}$$

empty set — complements — countable unions

Examples

discrete Countable: $\{r\} = \bigcap_{\varepsilon \in \mathbb{Q}^+} (r-\varepsilon, r+\varepsilon) \in \mathcal{B}_{\mathbb{R}}$

I countable $\Rightarrow$ $I = \bigcup_{r \in I} \{r\} \in \mathcal{B}_{\mathbb{R}}$

closed intervals: $[a,b] = (a,b) \cup \{a,b\}$

Non-examples?

More complicated: analytic, Lebesgue

Def: Measurable space $V = (\lfloor V \rfloor, B_V)$

Set (carrier)

family of subsets $B_V \subseteq P\lfloor V \rfloor$

closed under σ-algebra operations:

$$\frac{}{\emptyset \in B_V} \qquad \frac{A \in B_V}{A^c := \lfloor V \rfloor \setminus A \in B_V} \qquad \frac{\vec{A} \in B_V^{\mathbb{N}}}{\bigcup_{n=0}^{\infty} A_n \in B_V}$$

empty set

complements

countable unions

Idea: structure all spaces after the worst-case scenario

Examples

- Discrete spaces $\ulcorner X^{\text{meas}} \urcorner = (X, PX)$

- Euclidean spaces $\mathbb{R}^n$ — replace intervals with chests $\prod_{i=1}^{n} (a_i, b_i)$

  $\mathbb{R}^{\mathbb{N}}$ similarly

  $\{C \cap A \mid C \in B_V\}$

- Sub spaces: $A \in P\llcorner V \lrcorner$ $\quad A := (A, [B_V] \cap A)$

- Products: $A \times B := (\llcorner A \lrcorner \times \llcorner B \lrcorner, \sigma([B_A] \times [B_B]))$

Def: Borel measurable functions $f : V_1 \to V_2$

- functions $f : \lfloor V_1 \rfloor \longrightarrow \lfloor V_2 \rfloor$
- inverse image preserves measurability:

$$f^{-1}[A] \in \mathcal{B}_{V_1} \Longleftarrow A \in \mathcal{B}_{V_2}$$

Examples

- $(+), (\cdot) : \mathbb{R}^2 \longrightarrow \mathbb{R}$
- $|-|, \sin : \mathbb{R} \to \mathbb{R}$
- any continuous function $f : \mathbb{R}^n \to \mathbb{R}^m$
- any function $f : \ulcorner X \urcorner \to V$

Category Meas

Objects: Measurable spaces

Morphisms: Measurable functions

Identities:

$$id : V \to V$$

Composition:

$$\frac{f : V_2 \to V_3 \qquad g : V_1 \to V_2}{f \circ g : V_1 \to V_3}$$

Meas Category

Products, coproducts/disjoint union, Subspaces

Categorical limits, colimits, but:

Thm [Aumann '61] No σ-algebras $B_{B_{\mathbb{R}}}$, $B_{\mathbb{R}^{\mathbb{R}}}$ for measurable

membership predicate

$$(\ni) : (B_{\mathbb{R}}, B_{B_{\mathbb{R}}}) \times \mathbb{R} \longrightarrow \mathrm{Bool}$$

$$(U, r) \longmapsto [r \in U]$$

$$\mathrm{eval} : (\mathrm{Meas}(\mathbb{R}, \mathbb{R}), B_{\mathbb{R}^{\mathbb{R}}}) \times \mathbb{R} \to \mathbb{R}$$

$$(f, r) \mapsto f(r)$$

Questions? skip proof?

Proof (sketch):

Borel hierarchy:

$$\begin{tikzcd}[column sep=small]
\Sigma^0_0 \arrow[dr, phantom, sloped, "\subsetneq"] & & \Sigma^0_1 \arrow[dr, phantom, sloped, "\subsetneq"] & & & & \Sigma^0_\omega \arrow[dr, phantom, sloped, "\subsetneq"] & & \\
& \Delta^0_1 \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & & \Delta^0_2 \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & \cdots & \Delta^0_\omega \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & & \Delta^0_{\omega+1} \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & \\
\Pi^0_0 \arrow[ur, phantom, sloped, "\subsetneq"] & & \Pi^0_1 \arrow[ur, phantom, sloped, "\subsetneq"] & & & & \Pi^0_\omega \arrow[ur, phantom, sloped, "\subsetneq"] & &
\end{tikzcd}$$

stabilises at $\Delta^0_{\omega_1} = \mathcal{B}(\Sigma^0_0) = \Delta^0_{\omega_1+1}$

$$\mathrm{rank}\, A := \min\{\alpha < \omega_1 \mid A \in \Delta^0_\alpha\}$$

then for $B_{B_{\mathbb{R}}} = P(B_{\mathbb{R}})$

$$(\ni) : (B_{\mathbb{R}}, B_{B_{\mathbb{R}}}) \times \mathbb{R} \to \mathbb{R}$$

$$(U, r) \mapsto [r \in U]$$

$B_{V \times U} = B([B_V] \times [B_U])$

If measurable:

$$\alpha := \mathrm{rank}\left((\ni)^{-1}[\mathrm{true}]\right) < \omega_1$$

Take $A \in B_{\mathbb{R}}$, $\mathrm{rank}\, A > \alpha$

But:

$$\alpha < \mathrm{rank}\, A = \mathrm{rank}\,\langle A, -\rangle^{-1}\left[(\ni)^{-1}[\mathrm{true}]\right] \le \mathrm{rank}\left((\ni)^{-1}[\mathrm{true}]\right) \le \alpha$$

#

More details in Ex. B

Sequential Higher-order Structure:

I Countable: $\quad V^{\ulcorner I \urcorner} = \prod_{i \in I} V$

$\Longrightarrow$ Some higher-order structure in **Meas**:

$\mathrm{Cauchy} \in B_{[-\infty,\infty]^{\mathbb{N}}}$

$$\mathrm{Cauchy} := \bigcap_{\varepsilon \in \mathbb{Q}^+} \bigcup_{k \in \mathbb{N}} \bigcap_{\substack{m,n \in \mathbb{N} \\ m,n \geq k}} \{ \vec{y} \in [-\infty,\infty]^{\mathbb{N}} \mid |y_m - y_n| < \varepsilon \}$$

$\lim\sup : [-\infty,\infty]^{\mathbb{N}} \to [-\infty,\infty] \qquad \lim : \mathrm{Cauchy} \to \mathbb{R}$

Compose higher-order building blocks:

lim is measurable!

$$\mathrm{VanishingSeq}(\mathbb{R}) := \{ \vec{r} \in \mathbb{R}^{\mathbb{N}} \mid \lim_{n\to\infty} r_n = 0 \} \in B_{\mathbb{R}^{\mathbb{N}}}$$

$$\mathrm{approx}\_ : \mathrm{VanishingSeq}(\mathbb{R}^+) \times \mathbb{R} \longrightarrow \mathbb{Q}^{\mathbb{N}}$$

s.t.: $\left| \left(\mathrm{approx}_{\vec{\Delta}}\, r \right)_n - r \right| < \Delta_n$

Slogan: Measurable by Type !

Not all operations of interest fit:

Intrinsically higher-order !

$$\limsup : \left([-\infty,\infty]^{\mathbb{R}} \right)^{\mathbb{N}} \longrightarrow [-\infty,\infty]^{\mathbb{R}}$$

$$\limsup := \lambda \vec{f}.\lambda x. \limsup_{n\to\infty} f_n\, x$$

Want

Slogan: measurability by type!

But

For higher-order building blocks

defer measurability proofs until

we resume 1^{st} order fragment $\Rightarrow$ non compositional

Plan:

1) type-driven probability: discrete case (Mon + Tue)

2) Borel sets & measurable spaces (✓ Wed)

3) Quasi Borel spaces (Wed) Simple type structure (Thu)

4) Dependent type structure & standard Borel spaces (Thu)

5) Integration & random variables (Fri)

Please ask questions!

Course web page

scribble

Plan

Def: $V \in \mathbf{Meas}$ is Standard Borel when

$$V \cong A \quad \text{for some } A \in \mathcal{B}_{\mathbb{R}}$$

the "good part" of Meas — the subcategory

$$\mathbf{Sbs} \hookrightarrow \mathbf{Meas}$$

Sbs include

- Discrete $\ulcorner I \urcorner$, I countable
- Countable products of Sbs:

$$\mathbb{R}^n, \mathbb{R}^{\mathbb{N}}, 2^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}$$

- Borel subspaces of Sbs:

$$\mathbb{I} := [0,1] \qquad \mathbb{R}^+ := (0,\infty) \quad \mathbb{R}_{\geq 0} := [0,\infty]$$

- Countable coproducts of Sbs:

$$\mathbb{W} := [0,\infty] \qquad \overline{\mathbb{R}} := [-\infty,\infty]$$

Conservative extensions:

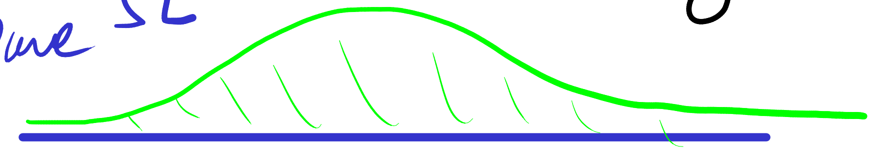
concrete spaces we "observe"

$$
\begin{tikzcd}
 & \text{Standard Borel spaces} \arrow[dl] \arrow[dr] & \\
\text{Meas} \arrow[dr] & & \text{Qbs} \arrow[dl] \\
 & \text{abstract (auxiliary?) spaces} &
\end{tikzcd}
$$

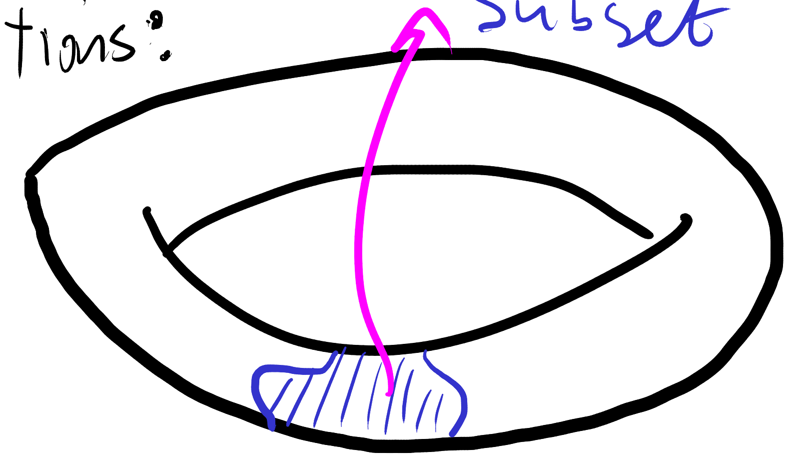
Core idea

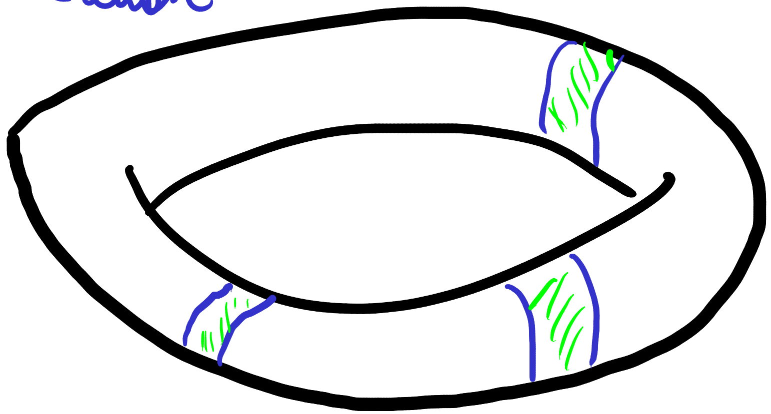
Measure Theory	Qbs Theory

sample space Ω



Primitive notions: measurable subset | random element $\downarrow \alpha$





Derived notions: random elements $\alpha : \Omega \to \text{Space}$ | measure | measurable subsets

Def: Quasi-Borel space $X = (\lfloor X \rfloor, R_X)$

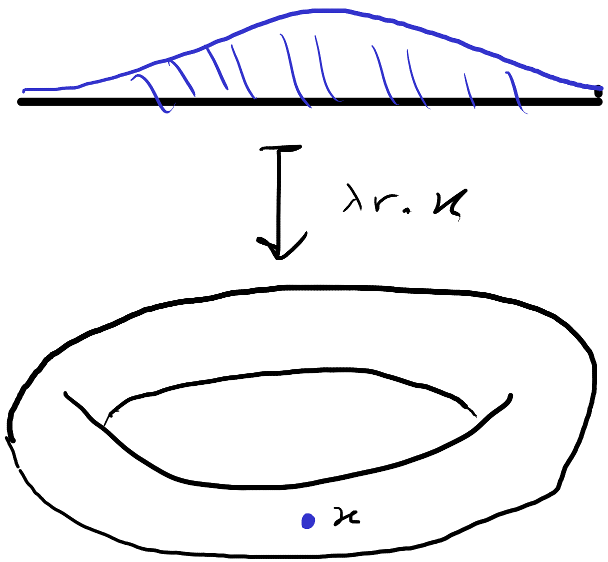
Set "carrier"

Set of functions $\alpha : \mathbb{R} \to \lfloor X \rfloor$ "random elements"

$R_X \subseteq \lfloor X \rfloor^{\lfloor \mathbb{R} \rfloor}$ closed under:

- Constants:

$$\frac{x \in \lfloor X \rfloor}{(\lambda r. x) \in R_X}$$



- precomposition:

- recombination

Def: Quasi-Borel space $X = (\lfloor X \rfloor, R_X)$

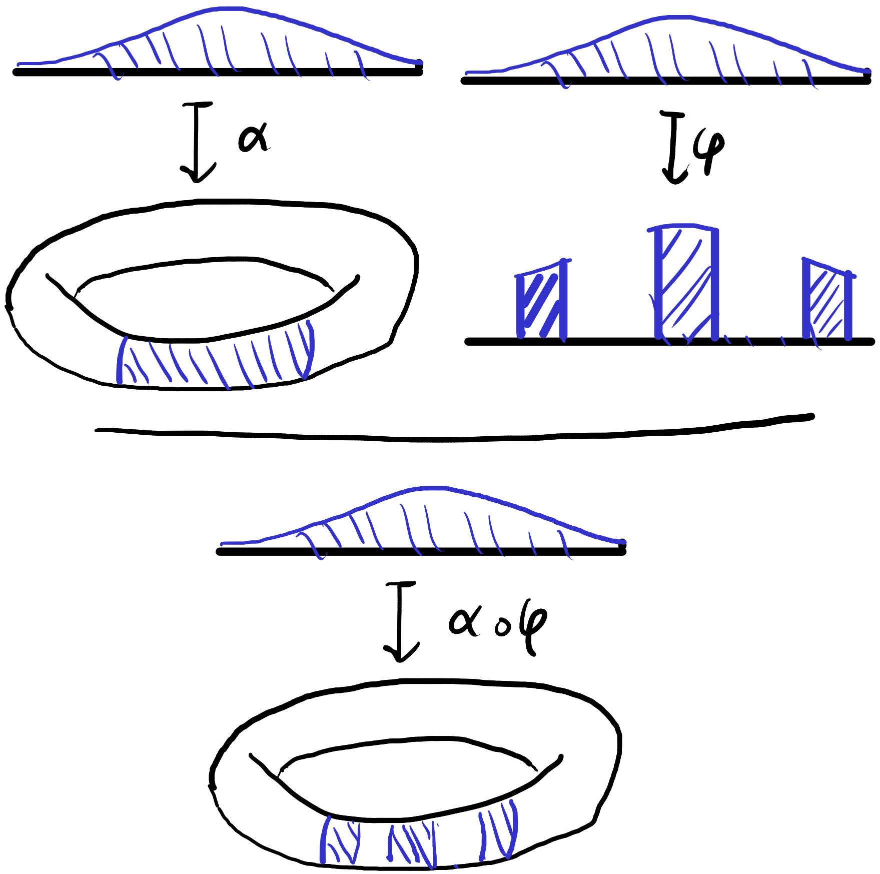
set "carrier"

set of functions $\alpha : \mathbb{R} \to \lfloor X \rfloor$ "random elements"

$R_X \subseteq \lfloor X \rfloor^{\lfloor \mathbb{R} \rfloor}$ closed under:

- precomposition:

$$\frac{\alpha \in R_X \qquad \varphi : \mathbb{R} \to \mathbb{R} \text{ in Sbs}}{\varphi \circ \alpha : \mathbb{R} \xrightarrow{\varphi} \mathbb{R} \xrightarrow{\alpha} \lfloor X \rfloor \in R_X}$$



Def: Quasi-Borel space $X = (\lfloor X \rfloor, R_X)$

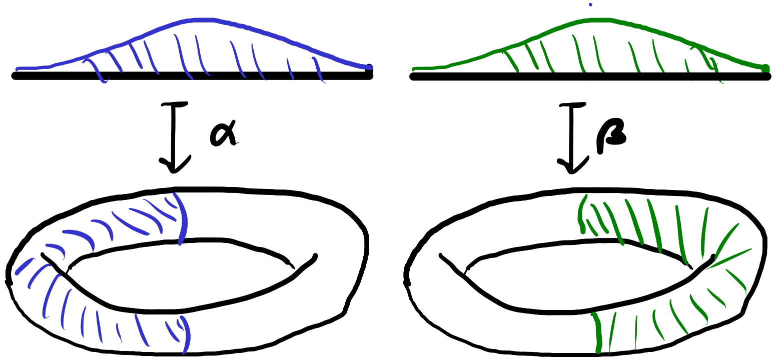
set "carrier"

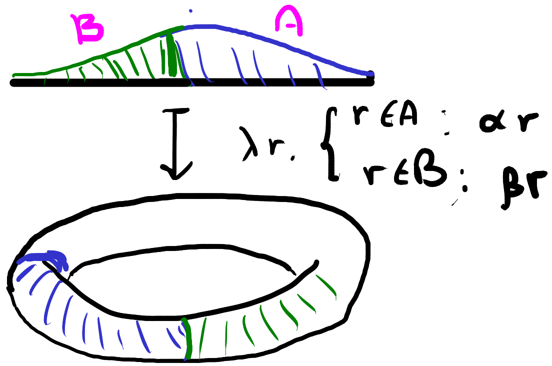
set of functions $\alpha : \mathbb{R} \to \lfloor X \rfloor$ "random elements"

$R_X \subseteq \lfloor X \rfloor^{\lfloor \mathbb{R} \rfloor}$ closed under:

- recombination

$$\frac{\vec{\alpha} \in R_X^{\mathbb{N}} \qquad \mathbb{R} = \biguplus_{n=0}^{\infty} \overbrace{A_n}^{\in B_{\mathbb{R}}}}{\lambda r. \begin{cases} \vdots \\ r \in A_n : \alpha_n r \\ \vdots \end{cases}}$$



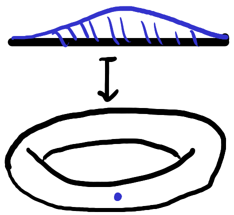


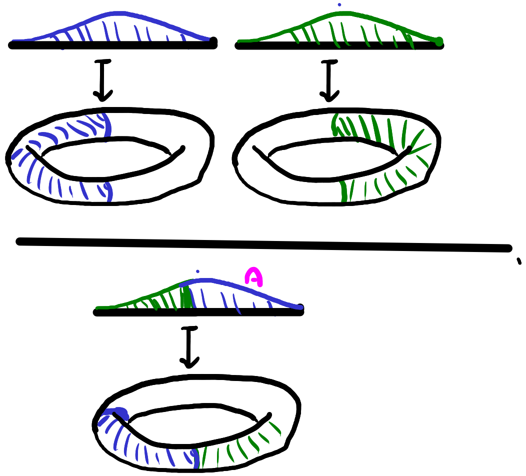
Def: Quasi-Borel space $X = (\lfloor X \rfloor, R_X)$

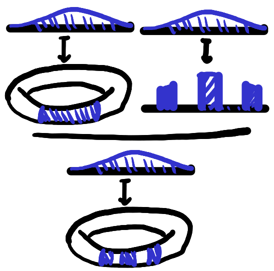
set "carrier"

set of functions $\alpha : \mathbb{R} \to \lfloor X \rfloor$ "random elements"

$R_X \subseteq \lfloor X \rfloor^{\lfloor \mathbb{R} \rfloor}$ closed under:

- Constants: 

- recombination 

- precomposition: 

Examples

recombination of constants

- $\mathbb{R} = (\lfloor\mathbb{R}\rfloor, \mathrm{Meas}(\mathbb{R},\mathbb{R}))$

qbs underlying $\mathbb{R}$

$$\lambda r. \begin{cases} \vdots \\ r \in A_n : x_n \\ \vdots \end{cases}$$

- $X \in \mathrm{Set}$, $\ulcorner X \urcorner^{\mathrm{Qbs}} := (X, \sigma\text{-simple}(\mathbb{R}, X))$

discrete qbs on X

- " $\lfloor X \rfloor_{\mathrm{Qbs}} := (X, X^{\lfloor\mathbb{R}\rfloor})$

all funtions

Indiscrete qbs on X

Qbs morphism $f : X \to Y$

- function $f : \lfloor X \rfloor \to \lfloor Y \rfloor$

$$- \quad \frac{\mathbb{R} \xrightarrow{\alpha} \lfloor X \rfloor \in R_X}{\mathbb{R} \xrightarrow{\alpha} \lfloor X \rfloor \xrightarrow{f} \lfloor Y \rfloor \in R_Y}$$

Example

- Constant functions are qbs morphisms
- σ-simple functions are qbs morphisms

Category Qbs $\Leftarrow$ - identity, composition

Full model

type : Qbs $\quad W := [0,\infty] \quad BX := $ (Thur)

$DX :=$ (Fri)

$PX := \{\mu \in DX \mid \mathcal{C}e_{\mu}[X] = 1\}$ (Thu)

$\mathcal{C}e_{\mu}[E] :=$ (Fri) $\qquad \delta_x :=$ (Fri)

$\oint \mu k :=$ (Fri)

Plan:

1) type-driven probability: discrete case (Mon + Tue)
2) Borel sets & measurable spaces (Wed)
3) Quasi Borel spaces (Wed) Simple type structure (Thu)
4) Dependent type structure & standard Borel spaces (Thu)
5) Integration & random variables (Fri)

Please ask questions!

Course web page

Scribble