Foundations for type-driven probabilistic modelling

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Partiality cf. [Vákár et al. '19]

A **Borel embedding** $e : X \hookrightarrow Y$

- injective function $e : \lfloor X \rfloor \rightarrowtail \lfloor Y \rfloor$
- its image is Borel: $e[\lfloor X \rfloor] \in \mathcal{B}_Y$
- e is **Strong**: $\alpha \in R_X \iff e \circ \alpha \in R_Y$

Examples

- $\mathbb{1} \hookrightarrow \mathbb{2}$
- S is sbs $\iff \exists\, S \hookrightarrow \mathbb{R}$

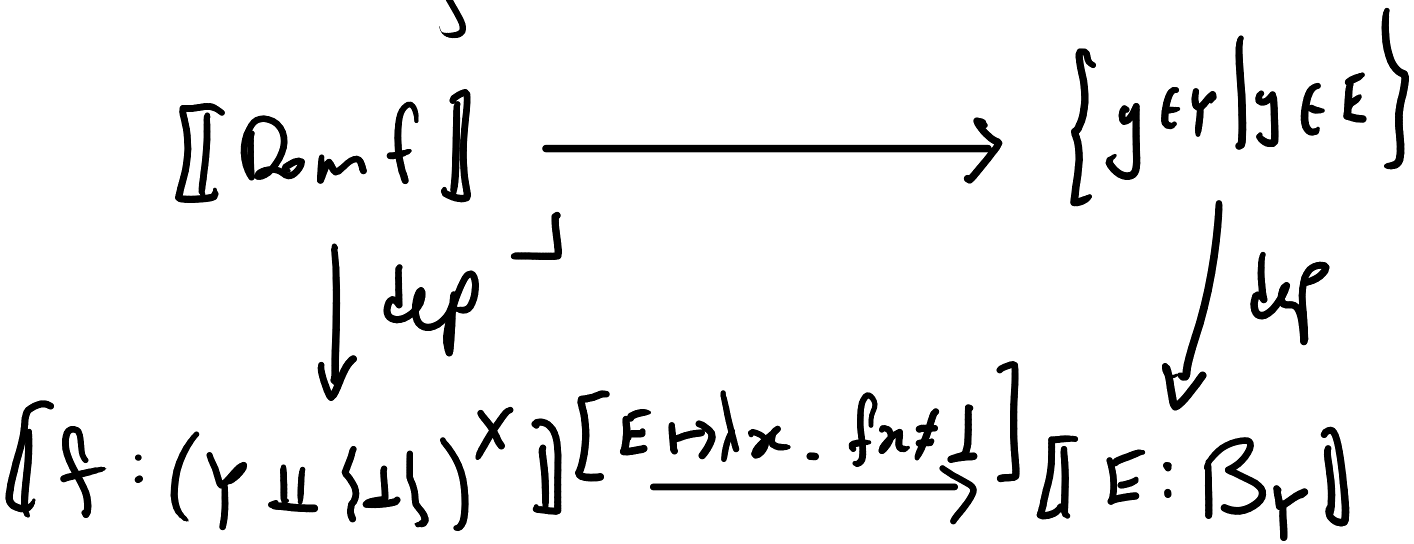
Def: A *partial map* $f: X \to Y$ is a morphism

$$f: X \to Y \amalg \{\bot\}$$

Its *domain of definition*

$$f:(Y \amalg \{\bot\})^X \vdash \mathrm{Dom}\, f := \{x \in X \mid fx \neq \bot\} : \mathrm{Type}$$

Dependent-type interpretation



Plan:

1) type-driven probability: discrete case (Mon + Tue)
2) Borel sets & measurable spaces (Wed)
3) Quasi Borel spaces (Wed) Simple type structure (Thu)
4) Dependent type structure & standard Borel spaces (Thu)
5) Integration & random variables (Fri)

Please ask questions!

Course web page

scribble

Full model

type : Qbs $\qquad \mathbb{W} := [0,\infty] \qquad \mathcal{B}X \cong \mathbb{B}^X$

$$DX := (\text{Fri})$$

$$PX := \{\mu \in DX \mid \mathcal{C}e_\mu[X] = 1\}$$

$$\mathcal{C}e_\mu[E] := (\text{Fri}) \qquad \delta_x := (\text{Fri})$$

$$\oint \mu k := (\text{Fri})$$

Def: A measure μ over $\mathbb{R}$ is a function

$$\mu : B_{\mathbb{R}} \to W := [0, \infty]$$

satisfying the measure axioms:

$$E : B^{W} \vdash$$

$$\mu \emptyset = 0, \quad \mu E = \mu(E \cap F) + \mu(E \cap F^{c}), \quad \mu\left(\bigcup_n E_n\right) = \sup_n \mu E_n$$

For measurable spaces, replace $\mathbb{R}$ with V

We write $\llcorner G V \lrcorner$ for the set of measures on V

For qbs X, take $\llcorner G \ulcorner X \urcorner^{\text{Meas}} \lrcorner$

Thm (Lebesgue measure):

There is a unique measure $\lambda \in \lfloor G\mathbb{R} \rfloor$ s.t.:

$\lambda(a,b) = b-a$

Thm (Lebesgue measure):

There is a unique measure $\lambda \in \lfloor G\mathbb{R} \rfloor$ s.t.:

$\lambda(a,b) = b-a$

Proof sketch (standard analysis textbook):

1) restrict attention to $(0,1]$ & extend via σ-additivity

2) Take $\Sigma_0 \subseteq B_{(0,1]}$ $E \in \Sigma_0 \Leftrightarrow E = \bigcup_{i=1}^{n} (a_i, b_i]$ ←

3) Defining $\lambda : \Sigma_0 \to W$, $\lambda \bigcup_{i=1} (a_i, b_i] := \sum_i (b_i - a_i)$ independent of '

4) $\lambda \phi = 0$, $\lambda E = \lambda(E \cap F) + \lambda(E \cap F^c)$ (straightforward)

↳

5) Technical gadget: $\forall (E_n \supseteq E_{n+1})$ in Σ_0,

$$\inf \lambda E_n > 0 \implies \bigcap E_n \neq \emptyset.$$

6) λ is continuous on Σ_0: If $(E_n \subseteq E_{n+1})_n$ in Σ_0

and $\bigcup_n E_n \in \Sigma_0$ then $\lambda \bigcup E_n = \sup_n \lambda E_n$

7) Noting that: Σ_0 is a Boolean algebra

& $\sigma(\Sigma_0) = \mathcal{B}_{(0,1]}$

we use Caratheodory's extension thm:

λ extends uniquely to $\lambda : \mathcal{B}_{(0,1]} \to \mathbb{W}$.

The unrestricted Giry spaces

Equip $\lfloor GV \rfloor$ with two qbs structures:

✗ $R_{GV} := \{\alpha : \mathbb{R} \to GV \mid \forall A \in B_V . \lambda r. \alpha(r, A) : \mathbb{R} \to W\}$

↳ α is a kernel.

✓ $GV \hookrightarrow W^{B_X}$ ↘

- Fewer random elements $R_{GV} \subseteq R_{G'V}$
- Lebesgue integral measurable in both arguments. (upcoming)

Farewell Meas

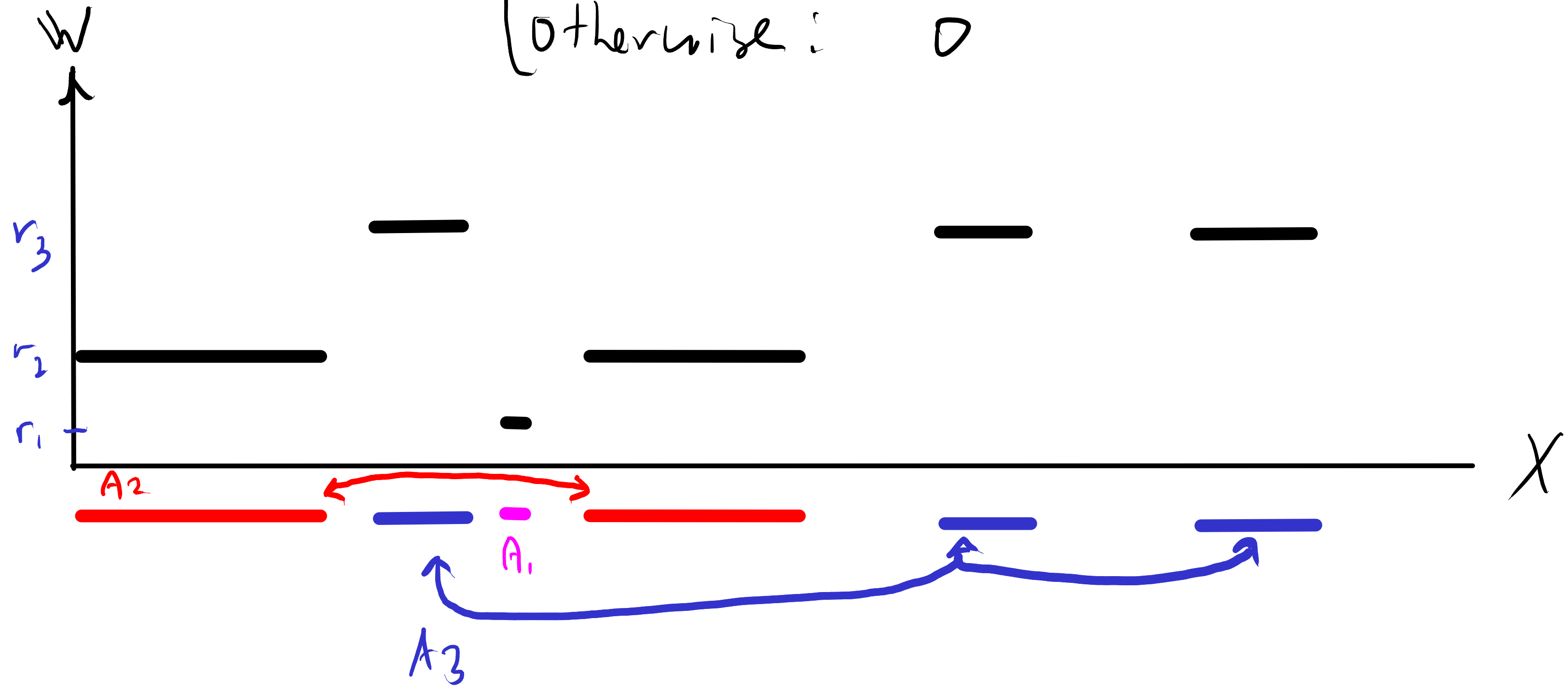
Now on:

1. All spaces are quasi-Borel
2. "measurable function" means qbs morphism!

Def: *Simple* function $\varphi : X \to W$ when

$\exists\, n \in \mathbb{N}$, $\vec{A} \in B_X^n$, $A_i \cap A_j = \emptyset$ $(i \neq j)$, $\vec{r} \in W$ s.t.

$$\varphi\, x = \begin{cases} \vdots & \\ x \in A_i : & r_i \\ \vdots & \\ \text{otherwise} : & 0 \end{cases}$$



Encode into a space:

$$\text{SimpleCode} := \coprod_{n \in \mathbb{N}} B_X^n \times W^n$$

$$\text{Simple} := \{ f \in W^X \mid f \text{ simple} \} \hookrightarrow W^X$$

and define an interpretation:

$$[\![-]\!] : \text{SimpleCode} \twoheadrightarrow \text{Simple}$$

$$[\![(n, \vec{A}, \vec{r})]\!] := \sum_{i=1}^{n} r_i \cdot [- \in A_i]$$

↳ characteristic function for A_i

Lemma: $f : X \to W$ is measurable → remember! Qbs morphism!

iff $f = \lim_{n\to\infty} f_n$ for some monotone sequence

$f_n \in \mathrm{Simple}$.

Moreover, we have measurable such choice:

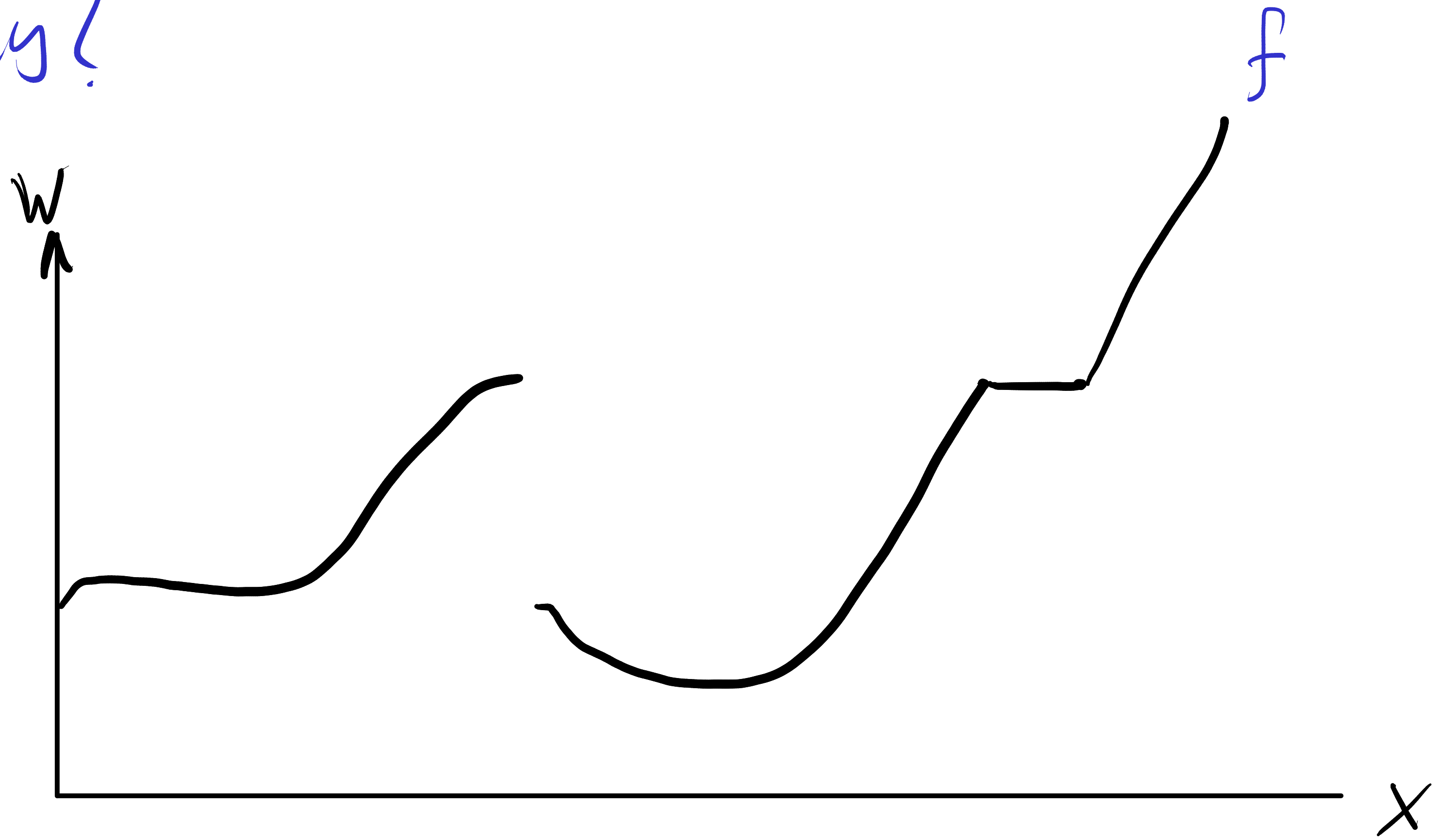
Simple Approx:

$$\{\vec{\Delta} \in \mathbb{R}^+ \mid \Delta_n \to 0\} \times \{\vec{a} \in W^{\mathbb{N}} \mid \vec{a} \text{ monotone}, a_n \to \infty\} \times W^X \to \mathrm{SimpleCode}$$

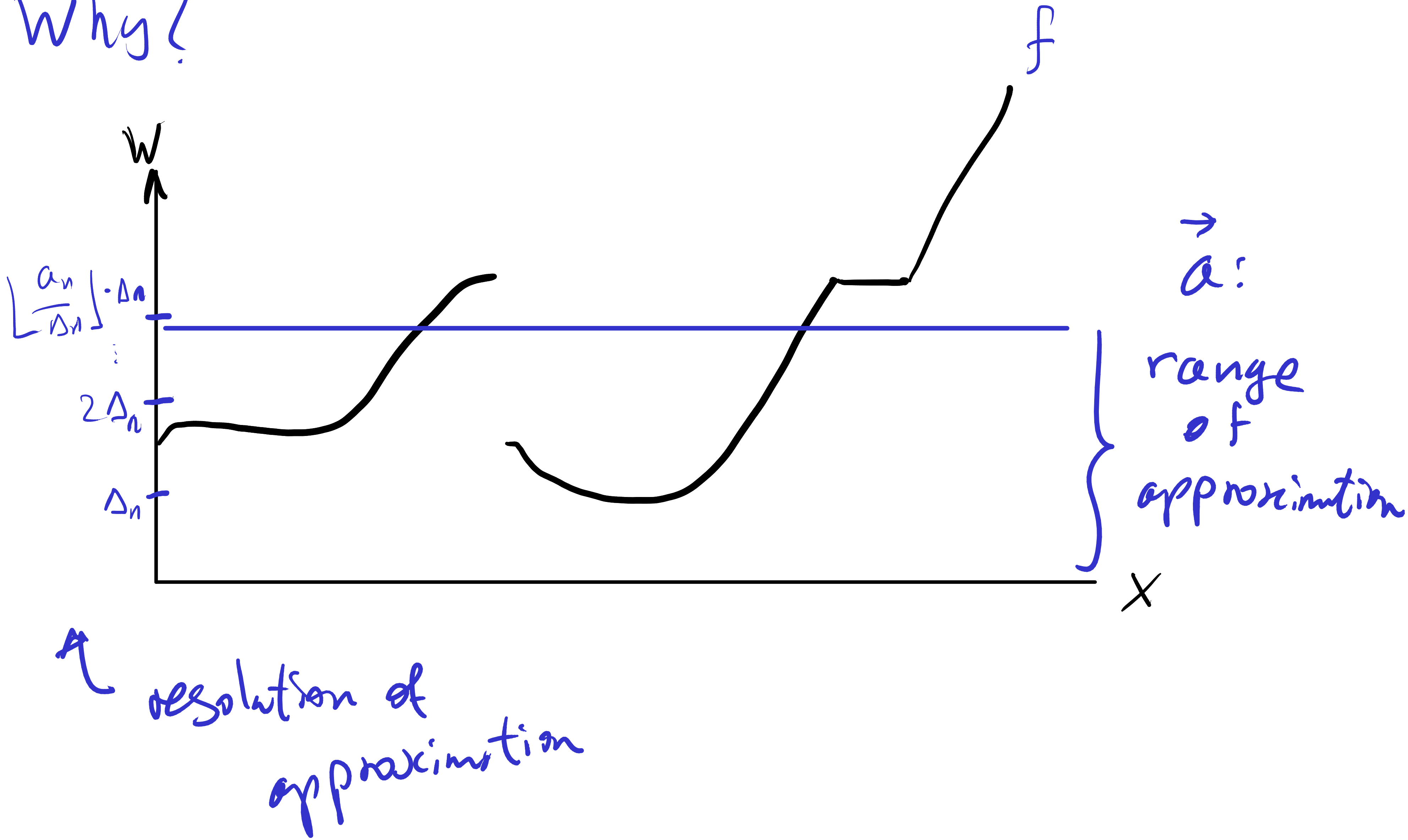
↑ rate of convergence

↑ range of approximation

Why?

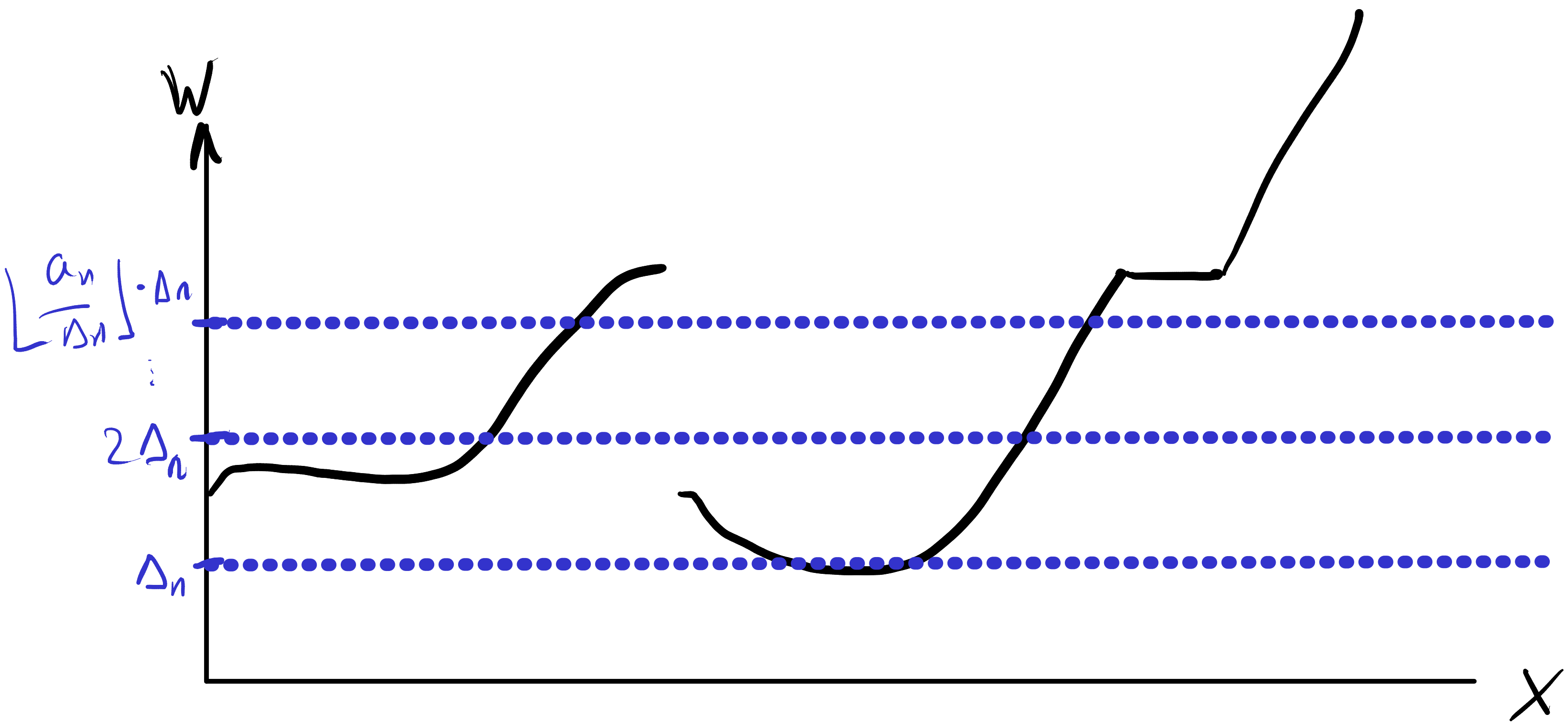


Why?



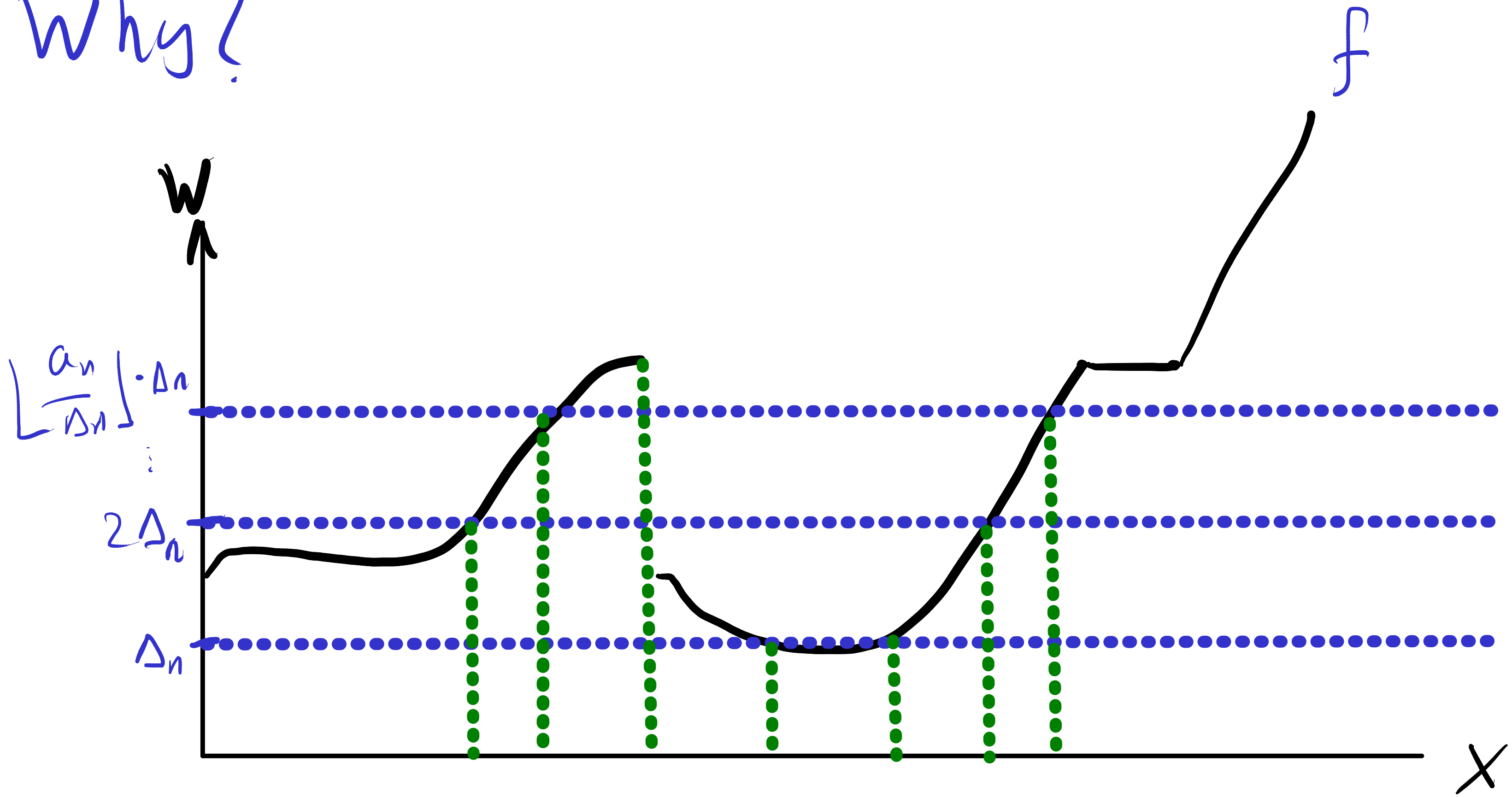
f
W
$\lfloor \frac{a_n}{\Delta_n} \rfloor \cdot \Delta_n$
$2\Delta_n$
Δ_n
X
$\vec{a}$:
range of f
approximation
resolution of approximation

Why?



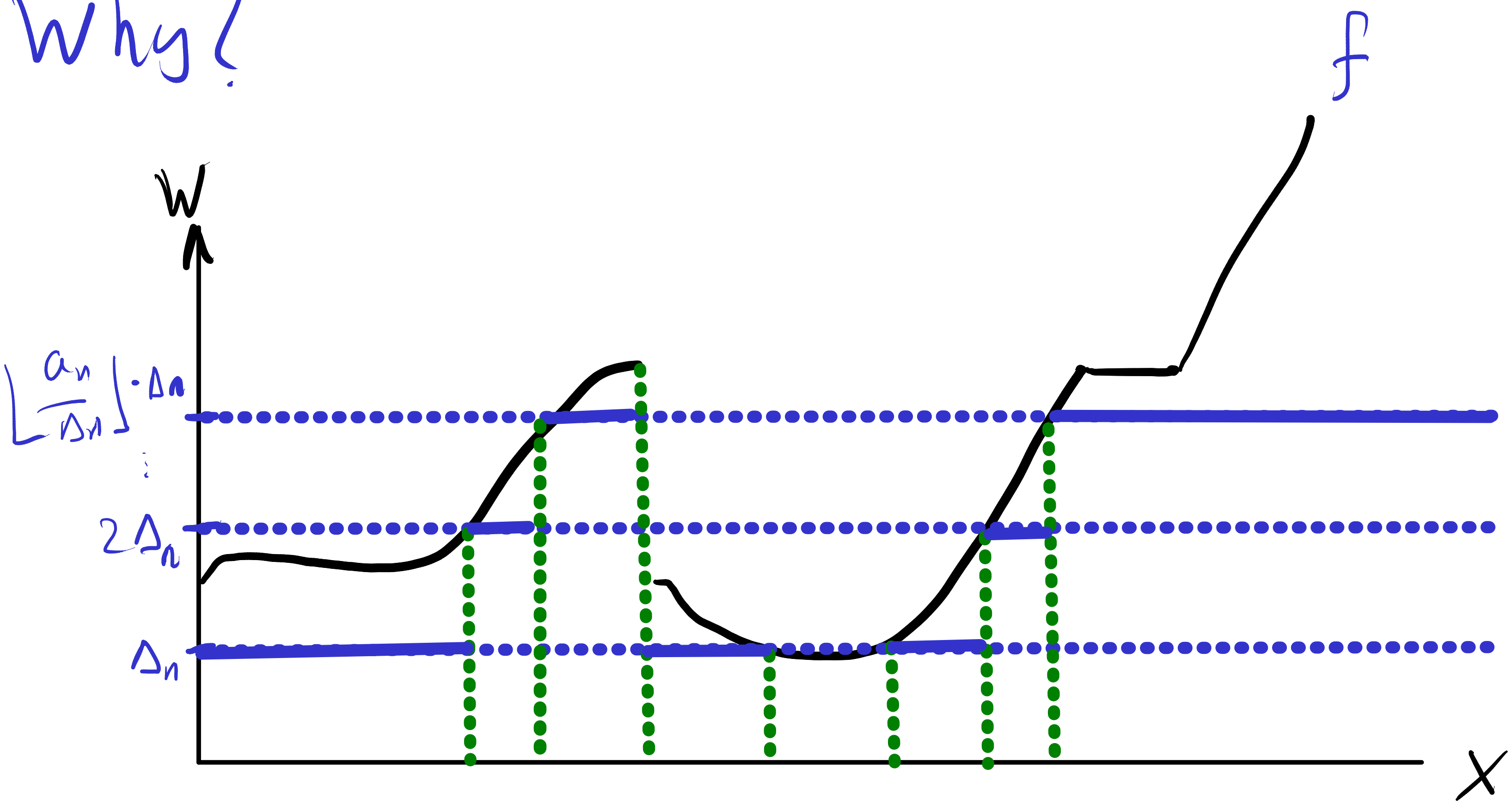
f
w
$\lfloor \frac{a_n}{\Delta_n} \rfloor \cdot \Delta_n$
$2\Delta_n$
Δ_n
X

Why?

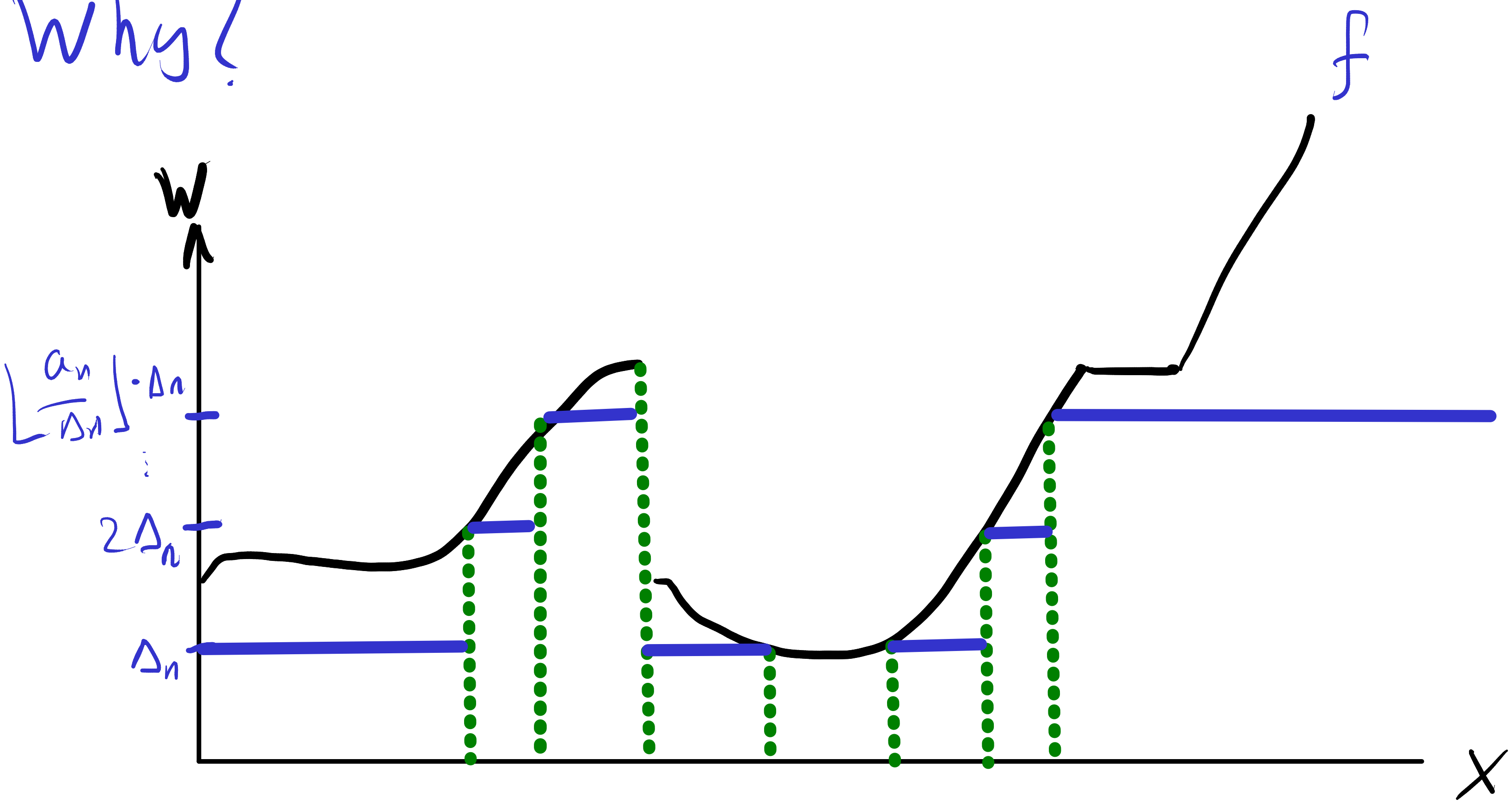


f
W
X
$\lfloor \frac{a_n}{\Delta_n} \rfloor \cdot \Delta_n$
$2\Delta_n$
Δ_n

Why?

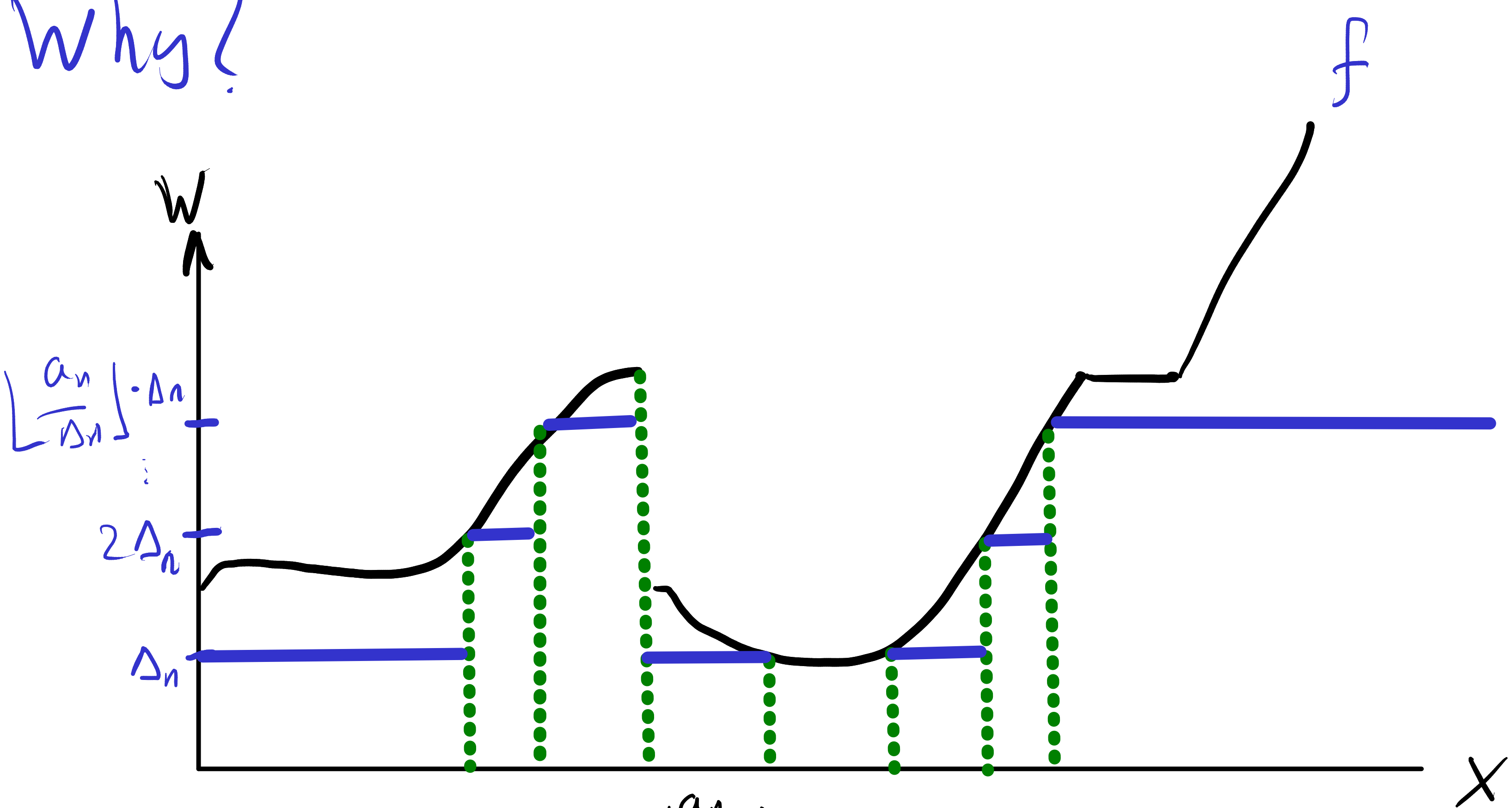

f
W
$\lfloor \frac{a_n}{\Delta_n} \rfloor \cdot \Delta_n$
$2\Delta_n$
Δ_n
X

Why?



f
W
X
$\lfloor \frac{a_n}{\Delta_n} \rfloor \cdot \Delta_n$
$2\Delta_n$
Δ_n

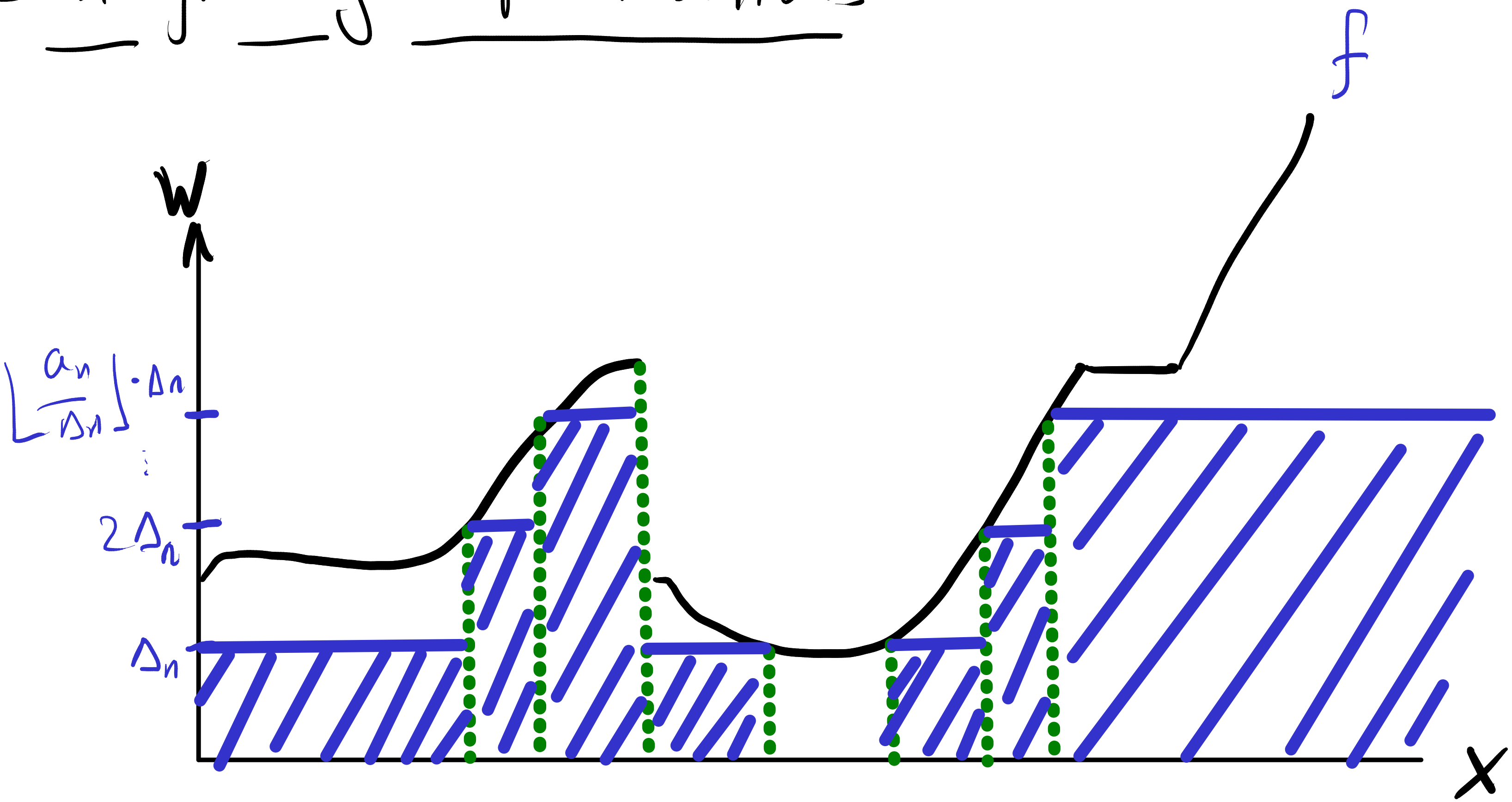
Why?



$$[\![\mathrm{SimpleApprox}_{\vec{\Delta},\vec{a}}\, f]\!] := \sum_{i=1}^{\lfloor \frac{a_n}{\Delta_n} \rfloor} i \cdot \Delta_n \left[i \cdot \Delta_n \le f < (i+1)\Delta_n \right]$$

$$+ \left\lfloor \frac{a_n}{\Delta_n} \right\rfloor \Delta_n \cdot \left[f \geq \left\lfloor \frac{a_n}{\Delta_n} \right\rfloor \cdot \Delta_n \right] \in \mathrm{Simple}$$

Integrating Simple Functions



$$\int : GX \times \mathrm{SimpleCode} \to W$$

$$\int \mu\,(n, \vec{A}, \vec{r}) := \sum_{I \subseteq \{1,\ldots,n\}} \Big(\sum_{i \in I} r_i\Big) \cdot \mu\Big(\bigcap_{i \in I} A_i \setminus \bigcup_{i \notin I} A_i\Big)$$

Integration

$$\int : GX \times W^X \longrightarrow W$$

Proper higher-order operation

$$\int \mu f := \sup \left\{ \int \mu \varphi \;\middle|\; \varphi \in \mathrm{Simple},\ \varphi \le f \right\}$$

$$= \lim_{n \to \infty} \int \mu \left(\mathrm{SimpleApprox}_{\vec{\Delta}, \vec{a}}\, f\right)_n$$

measurable by type

we also write $\int \mu(dx)\, t$ for $\int \mu\, (\lambda x. t)$

for $\frac{a_n}{\Delta_n} \to 0$, eg. $\Delta_n = \frac{1}{2^n}$ $a_n = n$.

(Δ_n: resolution)

The unrestricted Giry Strong Monad

Unlike the unrestricted Giry on Meas.

Dirac:

$$\delta : X \to GX$$

$$x \mapsto \lambda A. \begin{cases} x \in A : & 1 \\ x \notin A : & 0 \end{cases}$$

but: non-commutative

(Fubini fails, just like in Meas)

Kleisli extension / Kock integral:

$$\oint : GX \times GY^{X} \longrightarrow GY$$

$$\oint \mu f := \lambda A. \int \mu(dx)\, f(x; A)$$

Fubini-Tonelli fails

$\# \in G\mathbb{R}$ $\quad \#E := \begin{cases} E \text{ finite}: & |E| \\ \text{o.w.}: & \infty \end{cases}$

$\lambda \in G\mathbb{R}$ $\quad$ Lebesgue $\quad k : \mathbb{R} \times \mathbb{R} \to W \cong G\mathbb{1}$

$k(x,y) := [x = y]$

$$\oint \#(dr) \underbrace{\oint \lambda(dx)\, k(x,y)} = \oint \# \, \underline{0} = \underline{0} \cong 0$$

$y : \mathbb{R} \vdash \{\langle\rangle\} \mapsto \lambda\{y\} \cdot 1 + \lambda\{y\}^c \cdot 0 = 0$

$\ne$

$$\oint \lambda(dx) \underbrace{\oint \#(dr)\, K(x,y)} = \oint \lambda(dx)\, \delta_{\langle\rangle} \cong \infty$$

$x : \mathbb{R} \vdash \{\langle\rangle\} \mapsto \#\{x\} \cdot 1 + 0 = 1$

Randomisable measures monad

$D \rightarrowtail G$

$\lambda A. \int_{\mathrm{Dom}\,\alpha} \lambda(\mathrm{Dom}\,\alpha)$

$\lfloor DX \rfloor := \{ \lambda_\alpha \mid \alpha : \mathbb{R} \rightharpoonup X \}$

Lebesgue measure

$R_{DX} := \{ \lambda x.\, \lambda_{\alpha x} \mid \alpha : \mathbb{R} \times \mathbb{R} \rightharpoonup X \}$

$\delta : X \to DX \qquad \oint : D\Gamma \times (DX)^\Gamma \to DX$ lift along $D \rightarrowtail G$.

D validates our measure axioms including Fubini-Tonelli

$\mu \in DX,\ \nu \in DY \vdash$

$$\oint \mu(dx) \oint \nu(dy)\, \delta_{(x,y)} = \oint \nu(dy) \oint \mu(dx)\, \delta_{(x,y)} =: \mu \otimes \nu$$

Thm: For sbs S, PS, $D_{\leq 1}S$, $D_{<\infty}S \in \mathbf{Sbs}$

and agree with their counterparts on **Meas**.

$\lfloor DS \rfloor = \{\mu \mid \mu \text{ s-finite}\}$

$R_{DS} = \{K : \mathbb{R} \to GD \mid k \text{ s-finite kernel}\}$

See [Staton'16]

Open: Is there a counterpart to D in **Meas**?

More modestly, is $DS \in \mathbf{Sbs}$?

(Hypothesis: no)

Distribution Submonads

A measure space $\underline{\Omega} = (\Omega, \mu)$

is a qbs Ω with

$\mu \in DX$.

Similarly: - finite measure space

- (sub) probability space.

$$PX := \{\mu \in DX \mid \mu X = 1\}$$

$$\downarrow$$

$$P_{\leq 1} X := \{\mu \in DX \mid \mu X \leq 1\}$$

$$\downarrow$$

$$P_{<\infty} X := \{\mu \in DX \mid \mu X < \infty\}$$

$$\downarrow$$

$$DX$$

Full model

type : Qbs $\qquad W := [0,\infty] \qquad \mathcal{B}X \cong \mathbb{B}^X$

$$DX := \left(\{ \lambda_\alpha \mid \alpha : \mathbb{R} \to X \}, \{ \lambda r. \lambda_{\alpha(r,-)} \mid \alpha : \mathbb{R} \times \mathbb{R} \to X \} \right)$$

$$PX := \{ \mu \in DX \mid \mathrm{Ce}_\mu[X] = 1 \}$$

$$\mathrm{Ce}_\mu[E] := \mu E \qquad\qquad \delta_x := E \mapsto \begin{cases} x \in E : 1 \\ x \notin E : 0 \end{cases}$$

$$\oint \mu k := \lambda E. \int \mu(dx) k(x; E)$$

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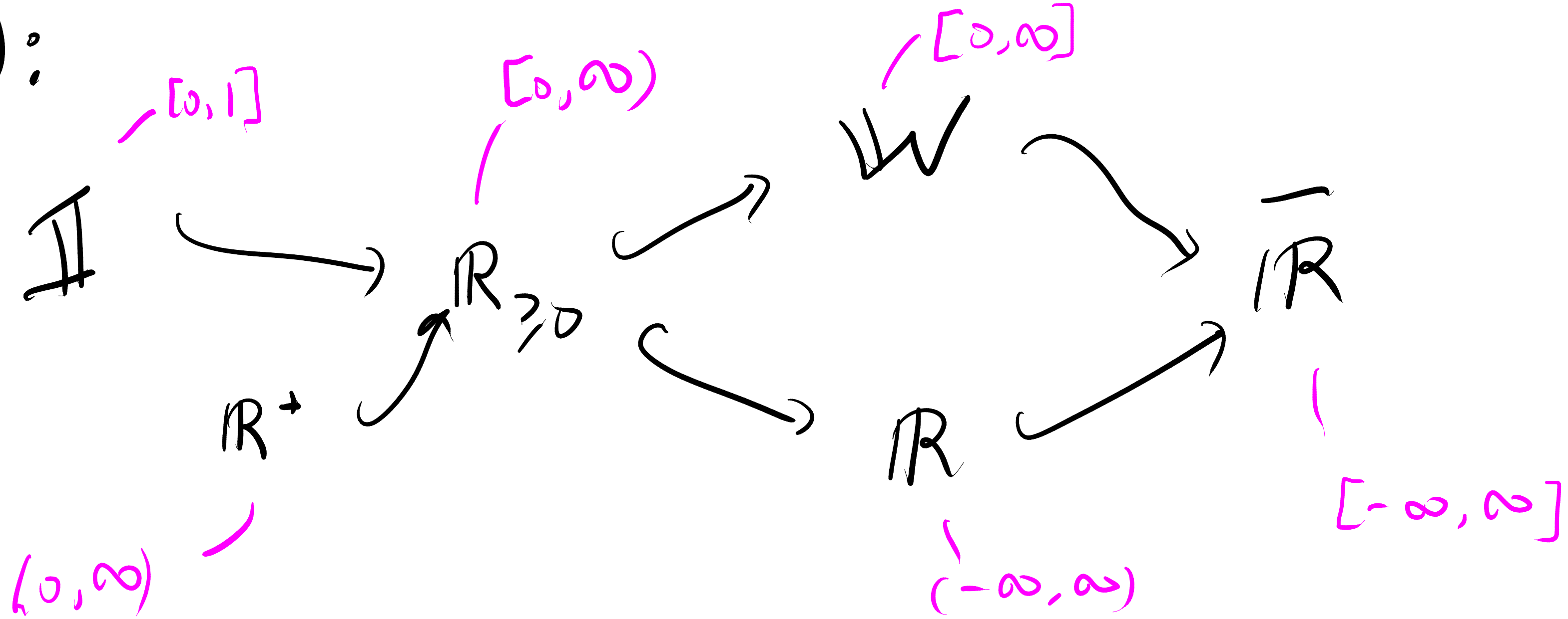
Scribble

Random variable: $\xi : \Omega \longrightarrow \Theta \hookrightarrow \overline{\mathbb{R}}$

Θ:



- Θ^{Ω} is a <u>space</u>
- $\mathbb{R}^{\Omega}$ measurable vector space:

$$\alpha\xi + \zeta := \lambda\omega.\ \alpha\cdot\xi\omega + \zeta\omega$$

- $\mathbb{W}^{\Omega}$ measurable σ-Semi-module for W: $\sum_{n=0}^{\infty} \alpha_n \xi_n :=$

$$\lambda\omega.\sum_{n=0}^{\infty} \alpha_n\cdot\xi_n\omega$$

$\Pr_- : P\Omega \times B_\Omega \to W$

$\Pr_\lambda A := eval(\lambda, A) = \lambda A$

Probability Space $\Omega = (\Omega, \lambda_\Omega)$

$P : P\Omega \vdash$ "$P\,x$ holds $\lambda(dx)$-almost surely"

for some $Q \hookrightarrow \Omega$, $P \supseteq Q$, $[- \in Q] \cdot \lambda = \lambda$

Example ($\xi, \zeta \in ℍ^\Omega$)

$\xi = \zeta$ a.s. when $\Pr_{\omega \sim \lambda}[\xi\omega \neq \zeta\omega] = 0$

Integrating random variables (as discretely) → in Qbs!

$(-)_+ , (-)_- : \overline{\mathbb{R}}^{\Omega} \longrightarrow W^{\Omega}$

$\xi_+ := \max(\xi, 0) \qquad \xi_- := \max(-\xi, 0)$

So: $\xi = \xi_+ - \xi_-$

$\int : P\Omega \times W^{\Omega} \longrightarrow W$

$\int \lambda\, \xi := \int \lambda\, \xi_+ - \int \lambda\, \xi_-$

$\int$ respects a.s. equality:

$\xi = \zeta \; (\lambda\text{-a.s.})$

$\Rightarrow \int \lambda\, \xi = \int \zeta.$

Example

$$\lambda : P\Omega \vdash \mathrm{ASConverge}(\overline{\mathbb{R}})^{\Omega} : \mathcal{B}\left(\overline{\mathbb{R}}^{\mathbb{N}\times\Omega}\right)$$

$$:= \left\{ \vec{\xi} \in \overline{\mathbb{R}}^{\mathbb{N}\times\Omega} \;\middle|\; \Pr_{\omega\sim\lambda}\left[\lim \xi_n \omega \neq \bot\right] \right\}$$

So:

$$\lim^{as} : \overline{\mathbb{R}}^{\mathbb{N}\times\Omega} \longrightarrow \overline{\mathbb{R}}^{\Omega} \qquad \mathrm{Dom}\,\lim^{as} := \mathrm{ASConverge}(\overline{\mathbb{R}})^{\Omega}$$

$$\lim^{as} \vec{\xi} := \lambda\omega.\ \limsup_{n\to\infty} f_n\,\omega$$

$\llcorner$ $\lim^{as}$ respects a.s. equality

Thm (monotone convergence):

Let $\vec{\xi} \in W^{\mathbb{N} \times \Omega}$ λ-a.s. monotone.

$$\xi = \lim_{n \to \infty}^{as} \xi_n \qquad \text{(a.s.)}$$

$$\Downarrow$$

$$\int \lambda\, \xi = \lim_{n \to \infty} \int \lambda\, \xi_n$$

Lebesgue space (Ω Prob. space, $p \in [1,\infty)$)

$$p : [1,\infty), \lambda : P\Omega \vdash \mathcal{L}^{p}_{(\Omega,\lambda)} : B(\mathbb{R}^{\Omega})$$

$$:= \left\{ \xi \in \mathbb{R}^{\Omega} \,\middle|\, \int \lambda |\xi|^{p} < \infty \right\} \hookrightarrow \mathbb{R}^{\Omega}$$

Ensemble $\mathcal{L}_{\Omega} := \prod_{\substack{\lambda \in P\Omega \\ p \in [1,\infty)}} \mathcal{L}^{p}_{(\Omega,\lambda)}$

$\llcorner$ $p \leq q \Rightarrow \mathcal{L}^{p}_{\Omega} \supseteq \mathcal{L}^{q}_{\Omega}$

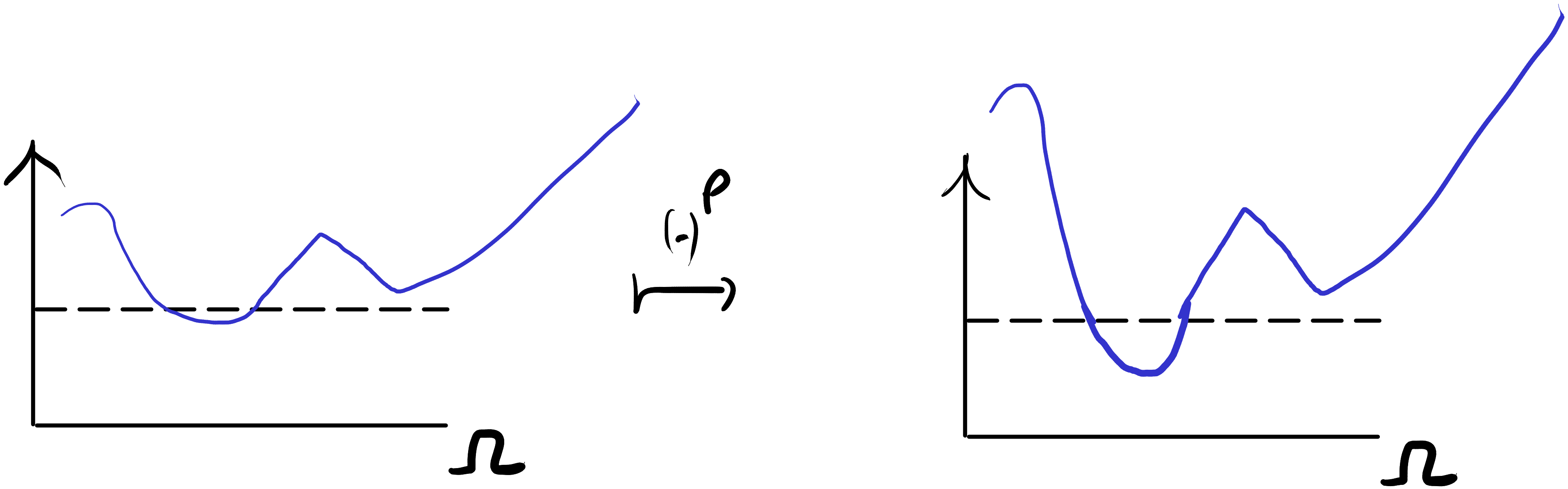
$\mathcal{L}^p$ semi norms

$$\|-\| : \prod_{p,\lambda} \mathcal{L}^p_{(\Omega,\lambda)} \to \mathbb{R}_{\geq 0} \qquad \|\xi\|_p := \sqrt[p]{\int \lambda |\xi|^p}$$

$\mathcal{L}^2$ inner product

$$\langle -, - \rangle : \prod_{p,\lambda} \mathcal{L}^p_{(\Omega,\lambda)} \times \mathcal{L}^p_{(\Omega,\lambda)} \to \mathbb{R}$$

$$\langle \xi, \zeta \rangle_{p,\lambda} := \int \lambda\, \xi \zeta$$



Statistics

Expectation

$$\mathbb{E} : \coprod_\lambda \mathcal{L}^1 \longrightarrow \mathbb{R}$$

$$\mathbb{E}_\lambda \xi := \int \lambda \xi$$

Covariance and Correlation

$$\mathrm{Cov}, \mathrm{Corr} : \coprod_\lambda \mathcal{L}^2 \to \mathbb{R}$$

$$\mathrm{Cov}(\xi, \zeta) := \langle \xi - \mathbb{E}\xi, \zeta - \mathbb{E}\zeta \rangle$$

$$\mathrm{Corr}(\xi, \zeta) := \frac{\langle \xi, \zeta \rangle}{\|\xi\|_2 \cdot \|\zeta\|_2} = \cos(\mathrm{angle}(\xi, \zeta))$$

Sequential limits

$p : [1,\infty), \lambda : PX_+$ $\text{Cauchy}\,\mathcal{L}^p_{(\Omega,\lambda)} : \mathcal{B}(\mathcal{L}^p_{(\mathbb{R},\lambda)})^{\mathbb{N}}$

$$:= \{ \vec{\xi} \mid \forall \varepsilon \in \mathbb{Q}^+ \, \exists k \in \mathbb{N} \, \forall m, n \geq k. \, \| \xi_{k+n} - \xi_{k+m} \|_p < \varepsilon \}$$

Thm: $\mathcal{L}^p_\Omega$ is Cauchy-Complete

$$\lim^p : \text{Cauchy}\,\mathcal{L}^p \longrightarrow \mathcal{L}^p$$ (convergence in mean)

Why?

1. Every Cauchy Sequence has an a.s. converging subseq.
2. We can find it measurably

Example

Thm (dominated convergence)

For $\vec{Z}, Z \in \mathcal{L}^1$ s.t. $Z_n \leq Z$ a.s.:

1. $\lim^{as} \vec{Z} \in \mathcal{L}^1$
2. $\lim^{1} \vec{Z} = \lim^{as} \vec{Z}$
3. $\lim_{n \to \infty} \int \lambda Z_n = \int \lambda \lim_{n \to \infty} Z_n$

Separability

Def: $\mathcal{L}^p$ separable: has countable dense subset

Fact: Separability is property of λ_Ω:

TFAE:

- $\exists p \geq 1.\ \mathcal{L}^p$ separable
- $\forall p \geq 1.\ \mathcal{L}^p$ separable

Measurable separability in $\mathcal{I} \hookrightarrow P\Omega \times [1,\infty)$

$$\vec{\beta} : \prod_{(\lambda,p)\in\mathcal{I}} \mathcal{L}^{p\,\mathbb{N}}_{(\Omega,\lambda)} \quad \text{s.t.}$$

$$\{\vec{\beta}^{\lambda,p}_n \mid n\in\mathbb{N}\} \text{ dense in } \mathcal{L}^p_{(\Omega,\lambda)}$$

Prop. - every sbs S measurably separable in

$$PS\times[1,\infty)$$

- $\mathcal{I} \hookrightarrow P\Omega \times \{2\}$ measurably separable

$$\Rightarrow \exists\ \vec{\beta} \in \prod_{\lambda\in\mathcal{I}} \mathcal{L}^2_{(\Omega,\lambda)} \quad \text{orthonormal system}$$

$\langle\beta_n,\beta_m\rangle = 0$

$\|\beta_n\|_2 = 1$

(β_n) dense

Example

Let $S \hookrightarrow \mathcal{L}^2$ closed vector subspace.

orthogonal decomposition — linear in fact.

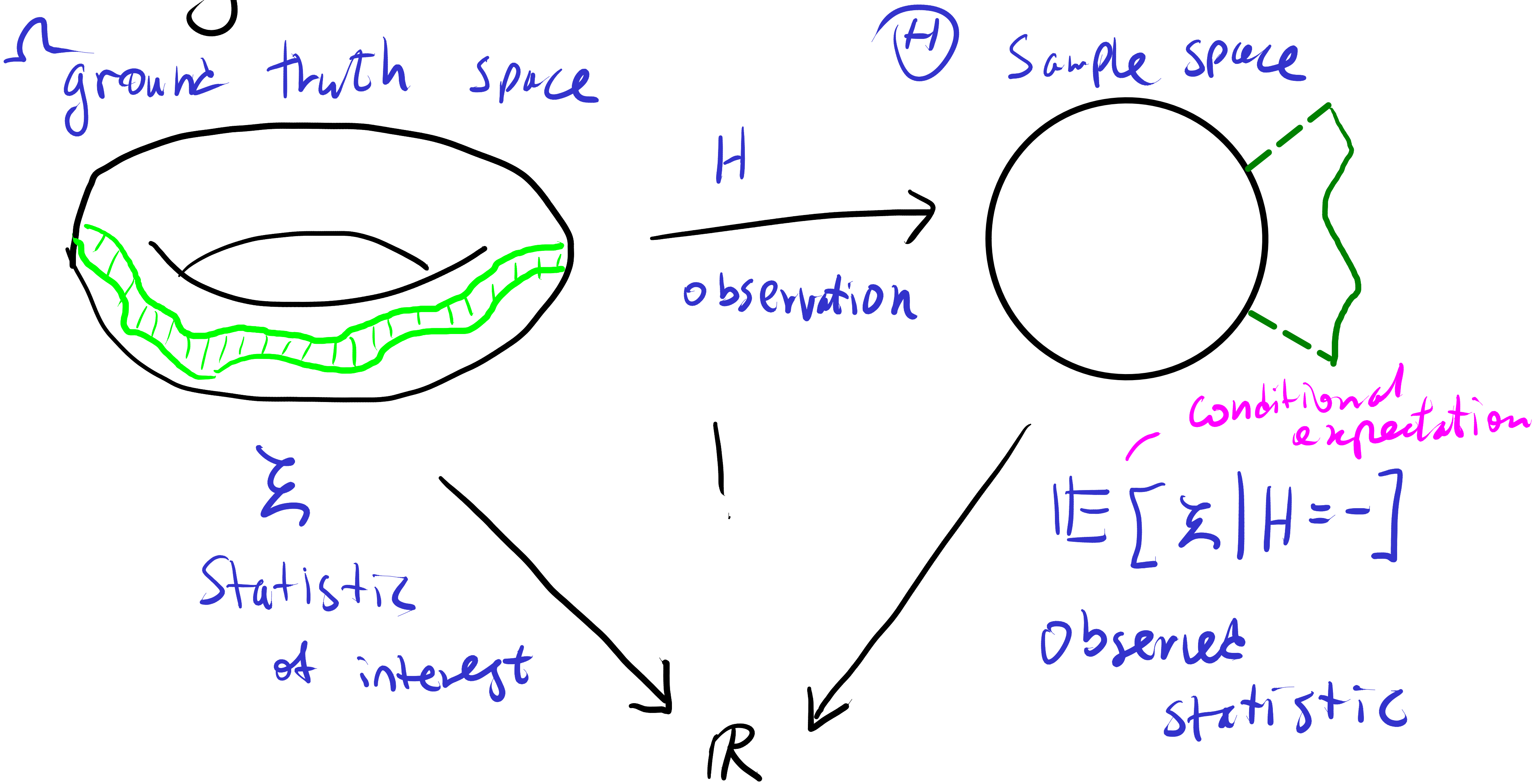
$$\langle P, P^{\perp} \rangle : \lfloor \mathcal{L}^P \rfloor \to \lfloor S \rfloor \times \lfloor S^{\perp} \rfloor$$

When S is separable with orthonormal system β

We have a measurable version of
$\langle P, P^{\perp} \rangle : \mathcal{L}^2 \to S \times S^{\perp}$

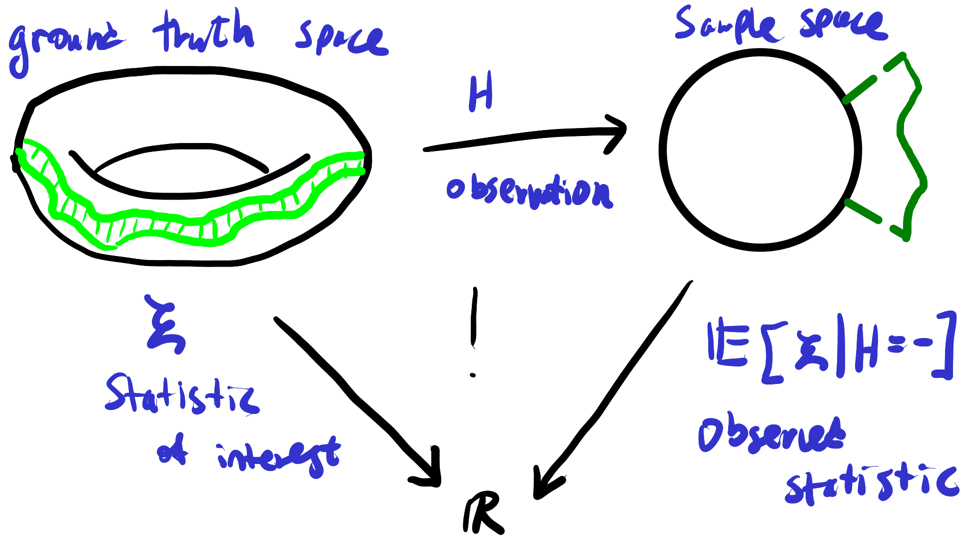
$$P\xi := \sum_{n=0}^{\infty} \langle \xi, \beta_n \rangle \beta_n \qquad P^{\perp} := \mathrm{Id} - P.$$

Kolmogorov's Conditional Expectation



Kolmogorov's Conditional Expectation

A conditional expectation of $\xi \in \mathcal{L}^1_\Omega$ wrt $H : \Omega \to \textcircled{H}$ is $Z \in \mathcal{L}^1_{\textcircled{H}}$ s.t. for all $A \in \mathcal{B}_{\textcircled{H}}$:



$$\int_A \mu Z = \int_{H^{-1}[A]} \lambda \xi$$

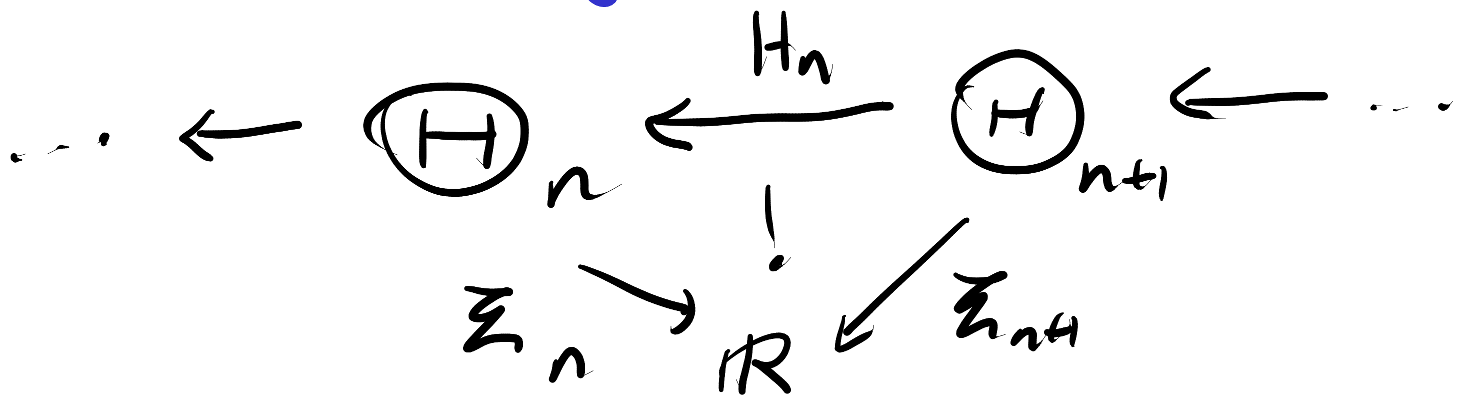
where $\mu := \lambda_H$

Conditional expectations

1. unique a.s.

2. fundamental to modern probability, e.g.:

   a martingale



s.t. $\Xi_n = \mathbb{E}[\Xi_{n+1} \mid H_n = -]$

Thm (Existence)

- $\exists \ \mathbb{E}[- \mid H = -] : \lfloor \mathcal{L}^1_{(\Omega,\lambda)} \rfloor \longrightarrow \lfloor \mathcal{L}^1_{(\Theta,\mu)} \rfloor$

- When (Ω,λ) is separable

$$\mathbb{E}[- \mid H = -] : \mathcal{L}^1_{(\Omega,\lambda)} \longrightarrow \mathcal{L}^1_{(\Theta,\mu)}$$

- When Θ is $\mathcal{I}'$-measurably separable

$$\mathbb{E}_{-}[- \mid - = -] : \prod_{\substack{H \in \Theta^{\Omega} \\ \lambda \in H_*^{-1}[\mathcal{I}]}} \mathcal{L}^1_{(\Omega,\lambda)} \longrightarrow \mathcal{L}^1_{(\Theta,\mu)}$$

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Discrete model

type : Set $\qquad \mathbb{W} := [0,\infty] \qquad \mathcal{B}X := \mathcal{P}X$

$$DX := \{\mu : X \to \mathbb{W} \mid \text{supp}\,\mu \text{ countable}\}$$

$$PX := \{\mu \in DX \mid \underset{\mu}{Ce}[X] = 1\}$$

$$\underset{\mu}{Ce}[E] := \sum_{x \in E} \mu x \qquad \delta_x := \lambda x'. \begin{cases} x = x' : & 0 \\ x \neq x' : & 1 \end{cases}$$

$$\oint \mu k := \lambda x. \sum_{\gamma \in \Gamma} \mu\gamma \cdot k(\gamma; x)$$

Full model

type : Qbs $\qquad \mathbb{W} := [0,\infty] \qquad \mathcal{B}X \cong \mathbb{B}^X$

$$DX := \left(\{\lambda_\alpha \mid \alpha : \mathbb{R} \to X\}, \{\lambda r.\lambda_{\alpha(r,-)} \mid \alpha : \mathbb{R}\times\mathbb{R} \to X\} \right)$$

$$PX := \{\mu \in DX \mid \mathop{Ce}_\mu [X] = 1\}$$

$$\mathop{Ce}_\mu [E] := \mu E \qquad \delta_x := E \mapsto \begin{cases} x \in E : 1 \\ x \notin E : 0 \end{cases}$$

$$\oint \mu k := \lambda E. \int \mu(dx) k(x;E)$$

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