4 Aumann's theorem

These exercises explore concepts derived from and around Aumann's theorem. We will not need intimate knowledge of the Borel hierarchy, but if you're curious about it, the exercises in Sec. A explore it in further detail through. This section is also an opportunity to learn and practice some category theory.

Let X, Y be measurable spaces. An *exponential of Y by X* is a pair $\langle Y^X, \mathrm{eval}\rangle$ consisting of a measurable space Y^X and a measurable function $\mathrm{eval} : Y^X \times X \to Y$ such that for every measurable space Γ and measurable function $f : \Gamma \times X \to Y$ there exists a unique measurable function $\lambda f : \Gamma \to Y^X$ satisfying:

$$\begin{array}{ccc} \Gamma & \quad & \Gamma \times X & & \\ {\scriptstyle \lambda f}\,\big\downarrow & & {\scriptstyle (\lambda f)\times \mathrm{id}_X}\,\big\downarrow & \overset{f}{\searrow} & \\ Y^X & & Y^X \times X & \xrightarrow[\mathrm{eval}]{=} & Y \end{array}$$

This definition is a standard category-theoretic notion — we could replace 'measurable space' by 'object' and 'measurable function' by 'morphism', as long as the category has products with X.

◸**4.1.** Let I be a countable set and Y a measurable space. Show that we can give an exponential of Y the discrete measurable space over I by the product $Y^{\ulcorner I \urcorner} := \prod_{i \in I} X$.

Where in your proof do you use I's countability? ◿

◸**4.2.** Let $\langle Y^X, \mathrm{eval}\rangle$ be an exponential in **Meas**.

- Find a bijection between the points in Y^X and the measurable functions from X to Y, that is: $\llcorner Y^X \lrcorner \cong \mathbf{Meas}(X, Y)$
- Show that there is a σ-algebra on $\mathbf{Meas}(X, Y)$ such that the set-theoretic evaluation function $\mathrm{eval} : \mathbf{Meas}(X, Y) \times X \xrightarrow{\langle f, x\rangle \mapsto f(x)} Y$ is measurable. ◿

◸**4.3.** Let I be a set.

Let $\langle X_i \rangle_{i \in I}$ be an I-indexed family of measurable spaces. Their *coproduct* $\langle \coprod_{i \in I} X_i, \iota_{-}\rangle$ consists of the measurable space $\coprod_{i \in I} X_i$ whose:

- points are pairs of a tag from I and a point from X_i:

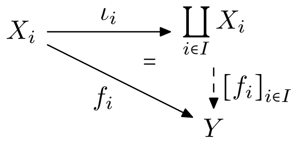
$$\llcorner \coprod_{i \in I} X_i \lrcorner := \coprod_{i \in I} \llcorner X_i \lrcorner := \bigcup_{i \in I} \{i\} \times \llcorner X_i \lrcorner$$

- measurable subsets are unions $\bigcup_{i \in I} \{i\} \times U_i$ of arbitrary I-indexed family of measurable subsets $U_i \in \mathcal{B}_{X_i}$.

and for each $i \in I$, $\iota_i : X_i \xrightarrow{x \mapsto \langle i, x\rangle} \coprod_{i \in I} X_i$.

Prove:

- $U \subseteq \coprod_{i \in I} \llcorner X_i \lrcorner$ is measurable iff $\iota_i^{-1}[U]$ is measurable for all $i \in I$.
- The σ-algebra axioms hold in the coproduct, and every injection is measurable.
- For every I-indexed family of measurable functions $f_i : X_i \to Y$ there is a unique measurable function $[f_i]_{i \in I} : \coprod_{i \in I} X_i \to Y$ such that:



- Find, and show the uniqueness of, the functorial action that makes the coproduct construction into a functor $\coprod_I : \mathbf{Meas}^I \to \mathbf{Meas}$ and all the coproduct injections natural transformations $\iota_i : \boldsymbol{\pi}_i \to \coprod_I$. ⊿

◸**4.4.** We say that a space X is *exponentiable* when there is an exponential Y^X for every measurable space Y.

- Let I be a set. Show that if, for every measurable space Γ, the following canonical map is a measurable isomorphism: $\coprod_{i\in I}\Gamma \xrightarrow{[\langle \mathrm{id}_\Gamma, i\rangle]_{i\in I}} \Gamma \times \ulcorner I \urcorner$, then $\ulcorner I \urcorner$ is exponentiable, and $\langle \prod_{i\in I} Y, \langle \vec{x}, i\rangle \mapsto x_i\rangle$ is the exponential $\left\langle Y^{\ulcorner I \urcorner}, \mathrm{eval}\right\rangle$ of Y by $\ulcorner I \urcorner$.
- Show, for every countable set I, that $\ulcorner I \urcorner$ is exponentiable.
- Show that if X is exponentiable, then for every I-indexed family of spaces, the canonical map $\coprod_{i\in I} X \to X \times \ulcorner I \urcorner$ is a measurable isomorphism. ⊿

Aumann's theorem shows that **Meas** cannot have an exponential for $\mathbb{R}$ by $\mathbb{R}$ by inspecting the full Borel hierarchy of the product. The next few exercises explore a more elementary example for two measurable spaces that don't have an exponential. I learned of this example from Christine Tasson and Johannes Hölzl.

◸**4.5.** Consider the following measurable spaces:

- $\ulcorner \mathbb{R} \urcorner := \langle \mathbb{R}, \wp\mathbb{R}\rangle$: the discrete measurable space over the real numbers.
- $\tilde{\mathbb{R}}$: the measurable space over the real numbers with the countable-cocountable σ-algebra.
- $\mathbb{2} := \mathbf{Fin}\,2 := \{\mathbf{true} := 1, \mathbf{false} := 0\}$: the discrete space with two points.

  We'll show that the exponential $\mathbb{2}^{\tilde{\mathbb{R}}}$ doesn't exist in **Meas**.

- Show that the diagonal $\{\langle r, r\rangle \in \mathbb{R} \times \mathbb{R} | r \in \mathbb{R}\}$ is a measurable subset of $\coprod_{r\in\mathbb{R}} \tilde{\mathbb{R}}$, and deduce that $\ulcorner \mathbb{R} \urcorner$ is not exponentiable.
  (This fact doesn't tell us which space Y doesn't have the exponential $Y^{\ulcorner \mathbb{R} \urcorner}$.)
- Show that if we have an exponential $\mathbb{2}^{\tilde{\mathbb{R}}}$, then the curried diagonal is a measurable function $\lambda r.\lambda s.[r = s] : \ulcorner \mathbb{R} \urcorner \to \mathbb{2}^{\tilde{\mathbb{R}}}$. ⊿

Aumann's theorem is still worth the effort. The spaces in the previous exercise may seem pathological, and we may falsely hope to exclude them by restricting to a subcategory of 'nice' spaces. Aumann's theorem concerns indispensable spaces: $\mathbb{2}$ and $\mathbb{R}$.

A frequent reaction to Aumann's theorem is to hope that we can avoid it by replacing the set of Borel measurable functions with a larger set of functions $f : \mathbb{R} \to \mathbb{R}$, such as the Lebesgue-measurable functions, or the universally measurable functions. This is not the case. Here's an 'easy', but unsatisfying, result:

◸**4.6.** Let E be a measurable space consisting of a σ-algebra over a set of functions that contains all the Borel measurable functions: $\mathbf{Meas}(\mathbb{R}, \mathbb{R}) \subseteq \llcorner E \lrcorner \subseteq \mathbf{Set}(\mathbb{R}, \mathbb{R})$. Show that the evaluation function $\mathrm{eval} : E \times \mathbb{R} \to \mathbb{R}$ is not measurable. ⊿

This result is unsatisfying because the σ-algebra on $\mathbb{R}$ in the domain of eval is the Borel one, so if we dare to include even one non-Borel-measurable function $f : \mathbb{R} \to \mathbb{R}$, then eval $\langle f, -\rangle : \mathbb{R} \to \mathbb{R}$ won't be measurable. The convincing result is that even if we take S to be the real numbers together with the much bigger σ-algebra of *Lebesgue*-measurable sets, then we still don't have any σ-algebra on the *Borel*-measurable functions that makes the evaluation function eval : $\mathbf{Meas}(\mathbb{R}, \mathbb{R}) \times S \to \mathbb{R}$ measurable. Doing so will require us to define the Lebesgue measurable sets, which will take us deeper into classical measure theory. This price is a hefty one to pay for just a dead-end, so I moved this material to Sec. B .If you're curious, jump right ahead.

References

Exercises