A The Borel hierarchy

These exercises concern the details behind the proof of Aumann's (1961) theorem. Flicking through, you'll see there's quite a lot to cover, but the rest of the material doesn't depend on this technical development. It's only here to satisfy your curiosity about what happens deep inside the σ-algebra of Borel sets. If you enjoy these, take a closer look at *descriptive set theory*. Two classical textbooks are Moschovakis's (1987) selection of key, central results, and Kechris's (1995) comprehensive, detailed, and slightly more modern book.

Define by transfinite induction on $\omega_1 + 1$, the successor of the first uncountable ordinal:

$$\boldsymbol{\Sigma}^{\mathcal{U}}_{\alpha}, \boldsymbol{\Pi}^{\mathcal{U}}_{\alpha}, \boldsymbol{\Delta}^{\mathcal{U}}_{\alpha} \subseteq \wp X \qquad (\alpha \in \omega_1)$$

$$\boldsymbol{\Sigma}^{\mathcal{U}}_{1} := \mathcal{U}$$

$$\boldsymbol{\Sigma}^{\mathcal{U}}_{\alpha+1} := \left\{ \bigcup_{i \in I} A_i \,\middle|\, I \subseteq \mathbb{N}, \vec{A} \in \mathcal{U} \cup \bigcup_{\beta \le \alpha} \boldsymbol{\Pi}^{\mathcal{U}}_{\beta} \right\} \qquad (1 \le \alpha \in \omega_1)$$

$$\boldsymbol{\Sigma}^{\mathcal{U}}_{\gamma} := \bigcup_{\beta < \gamma} \boldsymbol{\Sigma}^{\mathcal{U}}_{\beta} \qquad (1 \le \gamma \text{ a limit ordinal in } \omega_1)$$

$$\boldsymbol{\Pi}^{\mathcal{U}}_{\alpha} := \left[\boldsymbol{\Sigma}^{\mathcal{U}}_{\alpha}\right]^{\mathsf{C}} := \left\{ A^{\mathsf{C}} \,\middle|\, A \in \boldsymbol{\Sigma}^{\mathcal{U}}_{\alpha} \right\} \qquad \boldsymbol{\Delta}^{\mathcal{U}}_{\alpha} := \boldsymbol{\Sigma}^{\mathcal{U}}_{\alpha} \cap \boldsymbol{\Delta}^{\mathcal{U}}_{\alpha}$$

◸**A.1.** For every $\alpha \le \omega_1$, we have $\boldsymbol{\Sigma}^{\mathcal{U}}_{\alpha} \cup \boldsymbol{\Pi}^{\mathcal{U}}_{\alpha} \subseteq \boldsymbol{\Delta}^{\mathcal{U}}_{\alpha+1}$. ◿

◸**A.2.** Prove that $\sigma(\mathcal{U}) = \boldsymbol{\Sigma}^{\mathcal{U}}_{\omega_1} = \boldsymbol{\Pi}^{\mathcal{U}}_{\omega_1} = \boldsymbol{\Delta}^{\mathcal{U}}_{\omega_1}$. ◿

We therefore have the following relationships between the classes of the *Borel hierarchy*:

$$\begin{array}{ccccccccccccccc}
 & \boldsymbol{\Sigma}^{\mathcal{U}}_{1} & & \boldsymbol{\Sigma}^{\mathcal{U}}_{2} & & \boldsymbol{\Sigma}^{\mathcal{U}}_{3} & & & \boldsymbol{\Sigma}^{\mathcal{U}}_{\omega} & & \boldsymbol{\Sigma}^{\mathcal{U}}_{\omega+1} & & & \boldsymbol{\Sigma}^{\mathcal{U}}_{\omega_1} & \\
 \subseteq & & \subseteq\ \subseteq & & \subseteq\ \subseteq & & \subseteq & \subseteq & & \subseteq\ \subseteq & & \subseteq & = & & = \\
\boldsymbol{\Delta}^{\mathcal{U}}_{1} & & \boldsymbol{\Delta}^{\mathcal{U}}_{2} & & \boldsymbol{\Delta}^{\mathcal{U}}_{3} & & \cdots \subseteq & \boldsymbol{\Delta}^{\mathcal{U}}_{\omega} & & \boldsymbol{\Delta}^{\mathcal{U}}_{\omega+1} & & \cdots \subseteq & \boldsymbol{\Delta}^{\mathcal{U}}_{\omega_1} & & \sigma(\mathcal{U}) \\
 \subseteq & & \subseteq\ \subseteq & & \subseteq\ \subseteq & & \subseteq & \subseteq & & \subseteq\ \subseteq & & \subseteq & = & & = \\
 & \boldsymbol{\Pi}^{\mathcal{U}}_{1} & & \boldsymbol{\Pi}^{\mathcal{U}}_{2} & & \boldsymbol{\Pi}^{\mathcal{U}}_{3} & & & \boldsymbol{\Pi}^{\mathcal{U}}_{\omega} & & \boldsymbol{\Pi}^{\mathcal{U}}_{\omega+1} & & & \boldsymbol{\Pi}^{\mathcal{U}}_{\omega_1} &
\end{array}$$

Given a set V whose elements represent variables, the *σ-terms* over V are the countably-infinitary terms generated by the following grammar:

$$t, s ::= x \mid x^{\mathsf{C}} \mid \bigcup_{i \in I} t_i \mid \bigcap_{i \in I} t_i \qquad (x \in V, I \subseteq \mathbb{N})$$

Given a *valuation* $e : V \to \sigma(\mathcal{U})$, we can interpret each σ-term t as a Borel subset $[\![t]\!]\, e \in \sigma(\mathcal{U})$. Note that every term t involves only countably many variables, we call these variables its *support* suppt.

◸**A.3.** Let $\mathcal{U} \subseteq \wp X$, $\mathcal{V} \subseteq \wp Y$. Show that for every measurable $f : \langle X, \sigma(\mathcal{U})\rangle \to \langle Y, \sigma(\mathcal{V})\rangle$, the inverse image f^{-1} is a homomorphism of σ-terms:

$$f^{-1}[[\![t]\!]\, e] = [\![t]\!]\, (f^{-1} \circ e)$$ ◿

◸**A.4.** Show that if $e : V \twoheadrightarrow \mathcal{U}$ is surjective, then $[\![-]\!]\, e$ is surjective on $\sigma(\mathcal{U})$. ◿

We call a term *alternating* when, for every non-variable sub-term $f \langle t_i \rangle_i$, the root of each direct sub-tree is not the same operation symbol f.

◸**A.5.** Show that every term is denotationally equivalent to an alternating term. You might enjoy presenting a denotation-preserving terminating rewriting system. ◿

The *Aumann rank* function assigns to each Borel set the first stage in the hierarchy in which it occurs in some $\mathbf{\Sigma}^{\mathcal{U}}$ set:

$$\operatorname{rank}^{\mathcal{U}} : \sigma(\mathcal{U}) \to \omega_1$$
$$\operatorname{rank}^{\mathcal{U}} A := \min \left\{\alpha \in \omega_1 \middle| A \in \mathbf{\Sigma}^{\mathcal{U}}_{\alpha}\right\}$$

Define the *alternating depth* of a σ-term as follows:

$$\operatorname{alter} : \sigma\text{-Term}V \to \omega_1$$

$$\operatorname{alter} x := \operatorname{alter} x^{\complement} := 0 \quad \operatorname{alter} \bigcup_{i\in I} t_i := \bigvee_{i\in I} \operatorname{alter} t_i \quad \operatorname{alter} \bigcap_{i\in I} t_i := \bigvee_{i\in I} \operatorname{alter} t_i + \bigvee_{i\in I}\left[t_i \neq \bigcap_{j\in J} s_j\right]$$

◸**A.6.** Let t be a σ-term and e a valuation in some $\mathcal{U}$.

- Show that $[\![t]\!]\, e \in \mathbf{\Sigma}^{\mathcal{U}}_{\operatorname{alter} t \vee \alpha}$, where $\alpha := \bigvee_{x \in \operatorname{suppt}} e(x) \in \omega_1$.
- Deduce that if $e : V \to \mathcal{U}$, then $\operatorname{rank} [\![t]\!]\, e \leq \operatorname{alter} t$. Generalise to any $e : V \to \sigma(\mathcal{U})$.
- Show that $\operatorname{rank} A = \min \{\operatorname{alter} t | A = [\![t]\!]\, e\}$. ◿

◸**A.7.** Prove that if $A \in \sigma(\mathcal{U})$ and $\rho := \operatorname{rank}^{\mathcal{U}} A$, then:

$$A \cap \left[\mathbf{\Sigma}^{\mathcal{U}}_{\alpha}\right] \subseteq \mathbf{\Sigma}^{A\cap[\mathcal{U}]}_{\alpha} \subseteq \mathbf{\Sigma}^{\mathcal{U}}_{(\rho+1)\vee\alpha} \quad A \cap \left[\mathbf{\Pi}^{\mathcal{U}}_{\alpha}\right] \subseteq \mathbf{\Pi}^{A\cap[\mathcal{U}]}_{\alpha} \subseteq \mathbf{\Pi}^{\mathcal{U}}_{(\rho+1)\vee\alpha} \quad A \cap \left[\mathbf{\Delta}^{\mathcal{U}}_{\alpha}\right] \subseteq \mathbf{\Delta}^{A\cap[\mathcal{U}]}_{\alpha} \subseteq \mathbf{\Delta}^{\mathcal{U}}_{(\rho+1)\vee\alpha}$$ ◿

Let $\mathcal{U} \subseteq \wp X$, $\mathcal{V} \subseteq \wp Y$. When $\mathcal{V}$ is countable, define:

$$\operatorname{rank}^{\mathcal{U},\mathcal{V}} : \mathbf{Meas}(\langle X, \sigma(\mathcal{U})\rangle, \langle Y, \sigma(\mathcal{V})\rangle) \to \omega_1$$
$$\operatorname{rank} f := \bigvee_{A\in\mathcal{V}} f^{-1}[A]$$

Let $f : \langle X, \sigma(\mathcal{U})\rangle \to \langle Y, \sigma(\mathcal{V})\rangle$ be a measurable function.

◸**A.8.** What's the rank of a continuous function between two topological spaces? ◿

◸**A.9.** Bound the rank of $f^{-1}[A]$ for every $A \in \sigma(\mathcal{V})$, using $\operatorname{rank} A$ and $\operatorname{rank} f$.

Is your bound tight enough to deduce that $\operatorname{rank} f^{-1}[A] \leq \operatorname{rank} A$ when f is continuous for the topologies generated by $\mathcal{U}$ and $\mathcal{V}$? ◿

Let $\mathcal{U} \subseteq \wp(C \times X)$ and $\mathcal{V} \subseteq \wp X$ be two classes of subsets. We will regard subsets $[\![-]\!] \in \mathcal{U}$ as potential encodings for subsets in $\mathcal{V}$, where each element $c \in C$ encodes the section subset $[\![c]\!] := \{x \in X | x \in [\![c]\!]\}$.

We say that $[\![-]\!] \in \mathcal{U}$ is a *$\mathcal{U}$-$\mathcal{V}$-encoder* when $\mathcal{V} = \{[\![c]\!] | c \in C\}$. The intended meaning is that such an encoder lets us cover all the $\mathcal{V}$-subsets with a code in C. The literature uses the term *C-universal set for* $\mathbf{\Xi}$ for a $\mathcal{U}$-$\mathcal{V}$-encoder, when $\mathcal{U}$ and $\mathcal{V}$ belong to the same family of subset classes Ξ, such as $\mathcal{U} = \mathbf{\Sigma}_{\alpha}(C \times X)$ and $\mathcal{V} = \mathbf{\Sigma}_{\alpha}(X)$.

◸**A.10.** Show that if $[\![-]\!]$ is a $\mathcal{U}$-$\mathcal{V}$-encoder, then $[\![-]\!]^{\complement}$ is a $[\mathcal{U}]^{\complement}$-$[\mathcal{U}]^{\complement}$-encoder. ◿

◸**A.11.** Let $[\![-]\!]$ be a $\mathcal{U}$-$\mathcal{V}$ encoder, where $\mathcal{U} \subseteq \wp(C \times C)$ and $\mathcal{V} \subseteq \wp C$. Consider the diagonal function $\triangle := \lambda x. \langle x, x\rangle : C \to C \times C$.

Show that $\triangle^{-1}[[\![-]\!]^{\complement}] \notin \mathcal{V}$. ◿

We'll use this diagonalisation technique to show that the Borel hierarchy doesn't collapse for the reals.

◸**A.12.** Recall the Cantor space $\mathbb{G} \subseteq \mathbb{R}$, let $\mathcal{V}$ be the open subsets of $\mathbb{R}$, let $\mathcal{V}' := \mathbb{G} \cap [\mathcal{U}]$ be the open subsets in $\mathbb{G}$, and $\mathcal{U}'$ be the open subsets of $\mathbb{G} \times \mathbb{G}$.

- Show that if, for all $1 \le \alpha < \omega_1$, we have $\mathbf{\Sigma}_\alpha^{\mathcal{V}'} \neq \mathbf{\Pi}_\alpha^{\mathcal{V}'}$, then $\mathbf{\Sigma}_\alpha^{\mathcal{V}} \neq \mathbf{\Pi}_\alpha^{\mathcal{V}}$ too, and so the Borel hierarchy for $\mathbb{R}$ only stabilises at ω_1.
- Show that if $\mathbb{G}$ has a $\mathbf{\Sigma}_\alpha^{\mathcal{U}'}$-$\mathbf{\Sigma}_\alpha^{\mathcal{V}'}$-encoder, then $\mathbf{\Sigma}_\alpha^{\mathcal{V}'} \neq \mathbf{\Pi}_\alpha^{\mathcal{V}'}$.
- Show that, for all $1 \le \alpha \in \omega_1$, $\mathbb{G}$ has both a $\mathbf{\Sigma}_\alpha^{\mathcal{U}'}$-$\mathbf{\Sigma}_\alpha^{\mathcal{V}'}$ encoder and a $\mathbf{\Pi}_\alpha^{\mathcal{U}'}$-$\mathbf{\Pi}_\alpha^{\mathcal{V}'}$ encoder. ◿

The last exercise constructs a non-Borel set. This result doesn't fit the narrative, but we've already introduced most of the tools required for the job.

◸ **A.13.** A Borel set is *analytic* when it is empty, or a continuous image of the Baire space $\mathbb{Y} := \mathbb{N}^{\mathbb{N}}$. We denote by $\mathbf{\Sigma}_1^1(S)$ the class of analytic subsets of S. One can show that every Borel set is analytic, but that would require a lot of additional machinery.

- Show that if $\mathcal{B}_{\mathbb{Y}} \subseteq \mathbf{\Sigma}_1^1(\mathbb{Y})$ and we have a $\mathbf{\Sigma}_1^1(\mathbb{Y} \times \mathbb{Y})$-$\mathbf{\Sigma}_1^1(\mathbb{Y})$-encoder, then $\mathcal{B}_{\mathbb{Y}} \subset \mathbf{\Sigma}_1^1(\mathbb{Y})$.
- Show that we have a $\mathbf{\Pi}_1^0(\mathbb{Y} \times \mathbb{Y})$-$\mathbf{\Pi}_1^0(\mathbb{Y})$-encoder.
- Construct a homeomorphism $\mathbb{Y} \cong \mathbb{Y} \times \mathbb{Y}$. Derive a $\mathbf{\Pi}_1^0(\mathbb{Y} \times \mathbb{Y} \times \mathbb{Y})$-$\mathbf{\Pi}_1^0(\mathbb{Y} \times \mathbb{Y})$-encoder $\mathcal{F}[\![-]\!]$.
- Show that setting $x \in [\![c]\!]$ when $\exists z. \langle x, z\rangle \in \mathcal{F}[\![c]\!]$ is an $\mathbf{\Sigma}_1^1(\mathbb{Y} \times \mathbb{Y})$-$\mathbf{\Sigma}_1^1(\mathbb{Y})$-encoder.
  Hint: the graph of a continuous function over $\mathbb{Y}$ is a $\mathbf{\Pi}_1^0(\mathbb{Y} \times \mathbb{Y})$ set. ◿

References

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Exercises