1 Borel sets basics

Try these exercises if you're new to Borel sets of real numbers.

◸**1.1.** Show that the Borel sets are closed under:

- finite unions;
- countable intersections;
- translations:

$$A \in \mathcal{B}_{\mathbb{R}} \qquad \Longrightarrow \qquad r + [A] \coloneqq \{r + a | a \in A\} \in \mathcal{B}_{\mathbb{R}}$$ ◿

◸**1.2.** Show that the following sets are Borel ($a, b \in \mathbb{R}$):

- $[a, b]$;
- $\{a\}$;
- $(-\infty, a]$;
- $[a, b)$;
- $\mathbb{Q}$: the rational numbers ◿

Recall the *limit superior* and *limit inferior* operations on sequences of subsets $\vec{A} \subseteq X^{\mathbb{N}}$, thinking of them as subsets that vary in discrete time:

$\limsup_{n\to\infty} A_n \coloneqq \bigcap_{k\in\mathbb{N}} \bigcup_{\ell\geq k} A_\ell$: elements appearing *infinitely often* in the sequence;
$\liminf_{n\to\infty} A_n \coloneqq \bigcup_{k\in\mathbb{N}} \bigcap_{\ell\geq k} A_\ell$: elements appearing in *almost all* the sequence;
$\lim_{n\to\infty} A_n \coloneqq \liminf_n A_n = \limsup_n A_n$ when the two limits coincide.

If the elements of the sequence are Borel, so are the two limits.

For example, use sequences 3-valued indexed by natural numbers $\vec{x} \in \{0, 1, \text{wait}\}^{\mathbb{N}}$ to represent possibly-blocking streams of bits. Let $A_n \coloneqq \{\vec{x} | x_n \neq \text{wait}\}$. Then:

- $\limsup_n A_n$ are the streams that always produce more output; while
- $\liminf_n A_n$ are the streams that eventually stop blocking.

◸**1.3.** Practice manipulating limits of sets.

- (Taken from Wikipedia.) Calculate the two limits for the following sequences:
  - $\left\langle\left(-\frac{1}{n}, 1 - \frac{1}{n}\right)\right\rangle_n$
  - $\left\langle\left(\frac{(-1)^n}{n}, 1 - \frac{(-1)^n}{n}\right)\right\rangle_n$
  - $\left\langle\left\{\frac{i}{n} \middle| i = 0, \ldots, n\right\}\right\rangle_n$
- Show that:

$$\bigcap \vec{A} \subseteq \liminf \vec{A} \subseteq \limsup \vec{A} \subseteq \bigcup \vec{A}$$

- What happens to the two limits when $A_n \subseteq A_{n+1}$ and when $A_n \supseteq A_{n+1}$?
- This is the *indicator* function of a set $A \subseteq X$:

$$[- \in A] : X \to \{0, 1\}$$

$$[x \in A] \coloneqq \begin{cases} x \in A : & 1 \\ x \notin A : & 0 \end{cases}$$

Show that:

- $\bigcup \vec{A} = \{x \in X | \sup_n [x \in A_n] = 1\}$
- $\limsup \vec{A} = \{x \in X | \limsup_n [x \in A_n] = 1\}$
- $\liminf \vec{A} = \{x \in X | \liminf_n [x \in A_n] = 1\}$
- $\bigcap \vec{A} = \{x \in X | \inf_n [x \in A_n] = 1\}$ ⊿

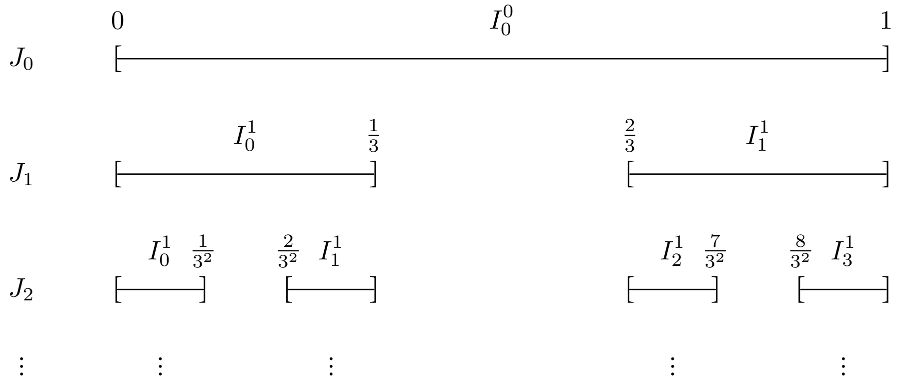
◸**1.4.** Let's construct the *Cantor set*. For each $n \in \mathbb{N}$, let $\mathbf{Fin}\, n := \{0, \ldots, n-1\}$ be the n-th cardinal. We define:

$$I : \coprod_{n=0}^{\infty} \mathbf{Fin}\, 2^n \to \left\{[a,b] \middle| b - a = \tfrac{1}{3^n}\right\} \subseteq \mathcal{B}_{\mathbb{R}}$$

as follows, writing $I_k^n := I(\iota_n k)$ for each $n \in \mathbb{N}$ and $k \in \mathbf{Fin}\, 2^n$:

$$I_0^0 := [0,1] \qquad I_{2k}^{n+1} := [\min I_k^{n+1}, \tfrac{1}{3^{n+1}} + \min I_k^{n+1}] \qquad I_{2k+1}^{n+1} := [\max I_k^{n+1} - \tfrac{1}{3^{n+1}}, \max I_k^{n+1}]$$

Each union $J_n := \bigcup_{k \in \mathbf{Fin}\, 2^n} I_k^n$ drops the middle thirds in the preceding interval sequence:



Later we'll define the *Lebesgue* measure as the unique σ-additive function $\boldsymbol{\lambda} : \mathcal{B}_{\mathbb{R}} \to [0, \infty]$ that assigns to each interval its length.

- Show that $\langle \boldsymbol{\lambda} J_n \rangle_n$ *vanishes*: $\lim_{n \to \infty} \boldsymbol{\lambda} J_n = 0$, by calculating each number in the sequence.
- The *Cantor set* is the limit $\mathbb{G} := \lim_n J_n$. Show that $\boldsymbol{\lambda} \mathbb{G} = 0$.
- Find a bijection $\mathbb{G} \cong \mathbb{T} := \mathbb{2}^{\mathbb{N}}$ where $\mathbb{2} := \mathbf{Fin}\, 2$.
- If you know some topology, equip $\mathbb{G} \hookrightarrow \mathbb{R}$ with the sub-space topology w.r.t. the open subsets of $\mathbb{R}$ and $\mathbb{T} = \prod_{n \in \mathbb{N}} \mathbb{2}$ with the product topology w.r.t. the discrete topology on $\mathbb{2}$. Find a homeomorphism $\mathbb{G} \cong \mathbb{T}$. ⊿

References