3 Basic category theory

We now have enough examples to introduce three important organising concepts from category theory: natural transformations, universal arrows, and adjunctions. This section is aimed at readers who want to take this opportunity to make first steps in category theory, but categorically-savvy readers might also learn some facts about the category of measurable spaces. There's too much material in this section for one sitting, so I recommend reading the first part of each subsection, and referring back to the more advanced parts if you need them later.

3.1 Natural transformations

Ex.2.15 constructs the product of two measurable spaces in the category of measurable spaces. We can record the fact that we can construct this product generally by organising products into a functor. The codomain of this functor is **Meas**, and its domain is the following.

◸**3.1.** Let $\mathbf{Meas}^2$ be the following category:

- Objects are pairs $\vec{X} = \langle X_1, X_2 \rangle$ of measurable spaces.
- Morphisms $\vec{f} : \vec{X} \to \vec{Y}$ are pairs of measurable maps between the corresponding spacse $\vec{f} = \langle f_1 : X_1 \to Y_1, f_2 : X_1 \to Y_1 \rangle$.

There's nothing specific about **Meas** here — we may as well replace it with two generic categories $\mathcal{C}_1, \mathcal{C}_2$ to construct the product category $\mathcal{C}_1 \times \mathcal{C}_2$.

- Spell out the objects and morphisms of $\mathcal{C}_1 \times \mathcal{C}_2$, define identities and composition, and show the resulting structure is a category.
- Define and prove functorial the two projection functors $\boldsymbol{\pi}_i : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_i$.
- Let $\mathcal{C}$ be a category. Define and prove functorial the *diagonal* functor $\boldsymbol{\Delta} : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$. ◿

Binary products organise into a functor $(\times) : \mathbf{Meas}^2 \to \mathbf{Meas}$:

- The action on objects maps each $\vec{X}$ to the binary product $X_1 \times X_2$.
- The action on morphisms maps each $\vec{f} : \vec{X} \to \vec{Y}$ to:

$$f_1 \times f_2 := \left\langle X_1 \times X_2 \xrightarrow{\pi_1} X_1 \xrightarrow{f_1} Y_1, \qquad X_1 \times X_2 \xrightarrow{\pi_2} X_2 \xrightarrow{f_2} Y_2 \right\rangle : X_1 \times X_2 \to Y_1 \times Y_2$$

(We apply this functor to pairs of objects and morphisms in infix notation.)

◸**3.2.** Show that $f_1 \times f_2$ is the unique measurable map satisfying for both $i = 1, 2$:

$$\begin{tikzcd} X_1 \times X_2 \arrow[r, "\pi_i"] \arrow[d, "f_1 \times f_2"'] \arrow[dr, phantom, "="] & X_i \arrow[d, "f_i"] \\ Y_1 \times Y_2 \arrow[r, "\pi_i"'] & Y_i \end{tikzcd}$$

◿

The equations in the previous exercise characterise the functorial action of the product, and the concept that organises them is that the projections $\pi_i^{\vec{X}} : X_1 \times X_2 \to X_i$ collect into a *natural transformation* $\pi_i : (\times) \to \boldsymbol{\pi}_i$.

In general, let $F, G : \mathcal{B} \to \mathcal{C}$ be functors. The structure of a natural transformation $\alpha : F \to G$, called a *transformation* from F to G is an assignment:

Exercises

- for each object $X \in \mathcal{B}$, a morphism $\alpha_X : FX \to GX$

The *naturality* property that makes a transformation a natural transformation is:

- for every morphism $f : X \to Y$ in $\mathcal{B}$, we have:

$$\begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] \arrow[dr, phantom, "="] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\alpha_Y"'] & GY \end{tikzcd}$$

◸**3.3.** Let $F, G : \mathbf{Meas}^2 \to \mathbf{Meas}$ are functors whose action on objects maps each $\vec{X}$ to the product $X_1 \times X_2$. Show that if both projections are natural, i.e., for each $i = 1, 2$:

$$\pi_i : F \to \boldsymbol{\pi}_i \qquad \pi_i : G \to \boldsymbol{\pi}_i$$

then F and G have the same action on morphisms. ⊿

◸**3.4.** Define the structure and prove the required properties of the following:

- The *identity* functor $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ for every category $\mathcal{C}$.
- The *diagonal* natural transformation $\Delta : \mathrm{Id}_{\mathbf{Meas}} \to (\times)$. ⊿

◸**3.5.** Let $\mathbf{Pred\,Meas} \hookrightarrow \mathbf{Meas}^2$ be the *subcategory* of $\mathbf{Set}^2$:

- Objects are those pairs $\vec{X}$ in which:
  - the points of X_1 are points in X_2: $\llcorner X_1 \lrcorner \subseteq \llcorner X_2 \lrcorner$
  - the σ-algebra on X_1 is the subspace σ-algebra we defined in Ex.2.8.

  So we have a measurable inclusion morphisms we write as $i : X_1 \hookrightarrow X_2$.
- Morphisms are those pairs $\vec{f} : \vec{X} \to \vec{Y}$ for which:

$$\begin{tikzcd} X_1 \arrow[r, hook, "i"] \arrow[d, "f_1"'] \arrow[dr, phantom, "="] & X_2 \arrow[d, "f_2"] \\ Y_1 \arrow[r, hook, "i"'] & Y_2 \end{tikzcd} \tag{1}$$

By stating it is a subcategory, we implicitly define the identities and composition in $\mathbf{Pred\,Meas}$ by the identities and composition in $\mathbf{Meas}^2$.

- Show that identities and composition are well-defined: identities satisfy the compatibility equation (1).
- Spell out the action of an inclusion functor $\mathbf{Pred\,Meas} \hookrightarrow \mathbf{Meas}^2$, and show it is indeed functorial, and moreover faithful.

  Since faithful functors reflect categories (Ex.2.6), $\mathbf{Pred\,Meas}$ is a category.
- Find functors $\mathrm{dom}, \mathrm{cod} : \mathbf{Pred\,Meas} \to \mathbf{Meas}$ that make the subspace inclusions into a natural transformation $i : \mathrm{dom} \to \mathrm{cod}$. ⊿

Let $\mathcal{B}, \mathcal{C}$ be categories. The category $\mathcal{C}^{\mathcal{B}}$ as functors as objects and natural transformations $\alpha : F \to G$ between them as morphisms.

◸**3.6.** Define identities and composition in $\mathcal{C}^{\mathcal{B}}$, faithful *evaluation* functors $\mathrm{eval}(-, X) : \mathcal{C}^{\mathcal{B}} \to \mathcal{C}$ for each $X \in \mathcal{B}$, and a faithful diagonal functor $\boldsymbol{\Delta} : \mathcal{C} \to \mathcal{C}^{\mathcal{B}}$. ⊿

◸**3.7.** Let $F, G : \mathcal{B} \to \mathcal{C}$ be functors. Show that a natural transformation $\alpha : F \to G$ is an isomorphism in $\mathcal{C}^{\mathcal{B}}$ iff each $\operatorname{eval}(\alpha, X) \coloneqq \alpha_X : FX \to GX$ is an isomorphism in $\mathcal{C}$. ◿

Every category structure $\mathcal{C}$ has an *opposite* category structure $\mathcal{C}^{\mathrm{op}}$ whose objects are the same, but a morphism from X to Y in $\mathcal{C}^{\mathrm{op}}$ is a morphism from Y to X in $\mathcal{C}$. We will never write morphisms $f : X \to_{\mathcal{C}^{\mathrm{op}}} Y$ in $\mathcal{C}^{\mathrm{op}}$ directly, but instead write them as $f : X \leftarrow Y$.

◸**3.8.** Show that a category structure $\mathcal{C}$ satisfies the defining properties of a category iff its opposite $\mathcal{C}^{\mathrm{op}}$ satisfies them. ◿

◸**3.9.** Let $\mathcal{C}$ be a category.

- Show that $f : X \to Y$ is an isomorphism in $\mathcal{C}$ iff $f : Y \leftarrow X$ is an isomorphism in $\mathcal{C}^{\mathrm{op}}$.
- Show that $\mathbb{1}$ is a terminal object of $\mathcal{C}$ iff $\mathbb{1}$ is an initial object of $\mathcal{C}^{\mathrm{op}}$. ◿

Category theorists use the adverb 'just' for this kind of process of unfolding all the structure and comparing the required properties of two concepts. So:

- an isomorphism in $\mathcal{C}^{\mathrm{op}}$ is just an isomorphism in $\mathcal{C}$;
- an initial object in $\mathcal{C}^{\mathrm{op}}$ is just a terminal object in $\mathcal{C}$;
- a natural isomorphism is just a natural transformation consisting of isomorphisms;
- $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ is just $\mathcal{C}$;

and so on. Unlike its colloquial usage, the technical meaning of 'just' doesn't imply this process is simple, obvious, or straightforward. Category theorists tend to forget this difference, which casual listeners sometimes find patronising. If you talk to someone who might not know the technical meaning of 'just', try using the more neutral 'amounts to'.

We define a *contravariant* functor F from $\mathcal{B}$ to $\mathcal{C}$ to be a functor $F : \mathcal{B}^{\mathrm{op}} \to \mathcal{C}$.

◸**3.10.** Show that contravariant functors:

- Reflect categories when faithful.
- Preserve isomorphisms.
- Reflect isomorphism pairs when faithful. ◿

◸**3.11.** A functor $H : \mathcal{B} \to \mathcal{C}$ is *fully-faithful* when its action on morphisms is bijective: for every morphism $g : HX \to HY$ there is a unique morphism $f : X \to Y$ such that $Hf = g$.

Show that fully-faithful functors *lift* isomorphic objects: if $H : \mathcal{B} \to \mathcal{A}$ is fully-faithful and $g : HA \xrightarrow{\cong} HB$ is an isomorphism, then there is an isomorphism $f : A \xrightarrow{\cong} B$ and H maps it to g. ◿

We'll now define the most important functor in category theory. Let $\mathcal{C}$ be a *locally small* category: each collection of morphisms from X to Y is a set $\mathcal{C}(X, Y)$ in our universe of sets. We then have the following functor $\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$:

- Its action on objects sends a pair of objects to the set of morphisms between them: $\mathrm{Hom}_{\mathcal{C}}\langle X, Y\rangle \coloneqq \mathcal{C}(X, Y)$.
- Its action on morphisms precomposes the contravariant argument and postcomposes the covariant argument:

$$\mathrm{Hom}_{\mathcal{C}}\langle f : X_1 \leftarrow X_2, g : Y_1 \to Y_2\rangle : \left(X_1 \xrightarrow{u} Y_1\right) \mapsto \left(X_2 \xrightarrow{f} X_1 \xrightarrow{u} Y_1 \xrightarrow{g} Y_2\right)$$

We'll write $\mathcal{C}(x, y)$ for $\mathrm{Hom}_{\mathcal{C}} \langle x, y\rangle$ for morphisms as well as objects. This notation matches previous conventions, like the product functor, where we used the same notation for morphisms and objects.

◸**3.12.** Show that $\mathrm{Hom}_{\mathcal{C}}$ is a functor. Show that its curried version $\mathbf{y}_{\mathcal{C}} : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ is also a functor. It is called the *Yoneda embedding*. Show that the alternative currying $\mathbf{y}' : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}^{\mathcal{C}}$ is just $\mathbf{y}_{\mathcal{C}^{\mathrm{op}}} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}^{(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}}$ for the opposite category. ◿

Because the iterated superscripts are hard to read, you'll see the notation $\hat{\mathcal{C}} := \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$.

◸**3.13.** Let $F : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$ be a functor from a *small* category $\mathbb{C}$: a category with a set of objects and a set of morphisms.

- Type-check that $\lambda x.\mathrm{Hom}_{\mathbf{Set}^{\mathbb{C}^{\mathrm{op}}}} \langle \mathbf{y}x, F\rangle : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$, which we may write as $\lambda x.\hat{\mathcal{C}}(\mathbf{y}x, F)$.
- Prove the *Yoneda lemma*: the operation 'evaluate each natural transformation at the identity morphism' is a natural isomorphism $\Upsilon : \left(\lambda x.\hat{\mathbb{C}}(\mathbf{y}x, F)\right) \xrightarrow{\cong} F$.
- Show that $\mathbf{y} : \mathbb{C} \to \hat{\mathbb{C}}$ is fully-faithful. ◿

3.2 Universality and representability

Universality lets us pin-point what makes a construction special. Let $H : \mathcal{B} \to \mathcal{C}$ be a functor, and $A \in \mathcal{C}$ an object. An *arrow from A to H* is a pair $\langle X, f\rangle$ consisting of:

- an object X in $\mathcal{B}$; and
- a morphism $f : A \to HX$ in $\mathcal{C}$.

An *arrow morphism* $h : \langle X, f\rangle \to \langle Y, g\rangle$ is a morphism $h : X \to Y$ satisfying:

$$\begin{tikzcd} & HX \arrow[d, "Hh"] \\ A \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[r, phantom, "="] & HY \end{tikzcd}$$

Arrows from A to H and their morphisms form a category. A *universal arrow from A to H* is an initial object in this category.

◸**3.14.** Define the remaining structure of the category of arrows from A to H. Define a faithful functor from this category structure to $\mathcal{B}$. ◿

◸**3.15.** Let A be a set. Find a universal arrow from A to the functor $\llcorner - \lrcorner : \mathbf{Meas} \to \mathbf{Set}$. ◿

◸**3.16.** Let V be a measurable space. Find a universal arrow from V to the functor $\mathrm{cod} : \mathbf{Pred}\,\mathbf{Meas} \to \mathbf{Meas}$ you defined in Ex.3.5. ◿

We define arrows from H to A similarly, as pairs $\langle X, f\rangle$ where $f : HX \to A$, and morphisms:

$$\begin{tikzcd} HX \arrow[d, "Hh"'] \arrow[dr, "f"] & \\ HY \arrow[r, "g"'] \arrow[r, phantom, shift left=3, "="] & A \end{tikzcd}$$

A universal arrow from H to A is then a terminal arrow in this category.

◸**3.17.** Find a universal arrow from the functor $\llcorner - \lrcorner : \mathbf{Meas} \to \mathbf{Set}$ to a set A. ◿

◸**3.18.** Find a universal arrow from the diagonal functor $\mathbf{\Delta} : \mathbf{Meas} \to \mathbf{Meas}^2$ to a pair of measurable spaces $\vec{X}$. ◿

◸**3.19.** Let A be a set. A *global geometry* $\mathcal{G}$ on A is a family of sets $\mathcal{G} \subseteq \wp A$. A *globally geometric space* X is then a pair $\langle \llcorner X \lrcorner, \mathcal{G}_X \rangle$ consisting of a set $\llcorner X \lrcorner$ of points and a global geometry $\mathcal{G}_X \subseteq \wp \llcorner X \lrcorner$. Given two globally geometric spaces X, Y, a *globally geometric morphism* $f : X \to Y$ is a function $f : \llcorner X \lrcorner \to \llcorner Y \lrcorner$ such that, for every subset in the codomain geometry $U \in \mathcal{G}_Y$, its inverse image is in the source geometry $f^{-1}[U] \in \mathcal{G}_X$.

- Define the structure of a category **Geom** whose objects are globally geometric spaces and their morphisms, and a faithful functor $\llcorner - \lrcorner : \mathbf{Geom} \to \mathbf{Set}$.
- Let A be a set. Find universal arrows from A to $\llcorner - \lrcorner$ and from $\llcorner - \lrcorner$ to A.
- Each σ-algebra is a global geometry, yielding a faithful functor $\llcorner - \lrcorner_{\mathbf{Geom}} : \mathbf{Meas} \hookrightarrow \mathbf{Geom}$. Let X be a globally geometric space. Find a universal arrow from $\llcorner - \lrcorner_{\mathbf{Geom}}$ to X. ◿

◸**3.20.** Let A be a set. Let $\mathbf{Rel}_A$ be the following category:

- objects are binary relations R over A, i.e.: $R \subseteq A \times A$; and
- there is a unique morphisms $f : R \to S$ when $R \subseteq S$.

Let $\llcorner - \lrcorner : \mathbf{Equiv}_A \hookrightarrow \mathbf{Rel}_A$ be the subcategory consisting of the equivalence relations and its associated faithful functor.

For every relation R, find a universal arrow from R to $\llcorner - \lrcorner$. ◿

Let I, $\mathcal{C}$ be categories. A *diagram of shape I in $\mathcal{C}$* is a functor $D : I \to \mathcal{C}$. A morphism $\alpha : D \to E$ between diagrams is a natural transformation. The functor category $\mathcal{C}^I$ then serves as the category of diagrams and their morphisms.

A *cone* for a diagram $D : I \to \mathcal{C}$ is a pair $\langle C, c \rangle$ consisting of:

- an object $C \in \mathcal{C}$, called the vertex of the cone; and
- a natural transformation $c : \mathbf{\Delta} C \to D$, i.e., an assignment for each $i \in I$ of a morphism $C \to Di$ in $\mathcal{C}$ such that for every $u : i \to j$ in I, we have:

$$\begin{tikzcd} & Di \arrow[d, "Du"] \\ C \arrow[ur, "c_i"] \arrow[dr, "c_j"'] \arrow[r, phantom, "="] & {} \\ & Dj \end{tikzcd}$$

A *cone morphism* $h : \langle B, b \rangle \to \langle C, c \rangle$ is a morhpism $h : B \to C$ satisfying, for all $i \in I$:

$$\begin{tikzcd} B \arrow[r, "h"'] \arrow[rr, bend left, "b_i"] & C \arrow[r, "c_i"'] & Di \end{tikzcd}$$

◸**3.21.** Show that a D-cone is just an arrow from the diagonal functor $\mathbf{\Delta} : \mathcal{C} \to \mathcal{C}^I$ to the diagram $D \in \mathcal{C}^I$. ◿

◸**3.22.** Find a category $\mathbf{2}$ so that the diagram category $\mathbf{Meas}^{\mathbf{2}}$ is just $\mathbf{Meas}^2$. ◿

A *limiting cone* is a universal cone, that is, a universal arrow from $\mathbf{\Delta}$ to D. Its vertex is called a *limit* of the diagram. Similarly, a *colimiting cocone* is a universal arrow from D to $\mathbf{\Delta}$, and its vertex is called the *colimit* of the diagram.

◸**3.23.** Show that a terminal object is just a limiting cone for the diagram from the category with no objects and no morphisms. ◿

◸**3.24.** Show that a limit in $\mathbf{Rel}A$ is just the intersection of the relations in the diagram, and a colimit is just the union. ◿

◸**3.25.** Let I be the category with two objects 0, 1 and four morphisms:

- The two identities: id_0, id_1; and
- $f : 0 \to 1$ and $g : 1 \to 0$.

Define composition to satisfy the neutrality axioms whenever an identity is involved, and in the remaining cases define:

$$f \circ g := \mathrm{id}_1 \qquad g \circ f := \mathrm{id}_0$$

- Define a faithful functor $U : I \to \mathbf{Set}$ sending 0 to $\{0\}$ and 1 to $\{1\}$ and deduce I is indeed a category.
- Show that a diagram $D : I \to \mathcal{C}$ is just an isomorphism pair. ◿

◸**3.26.** Let $D : I \to \mathbf{Set}$ be a *small* diagram — a diagram whose domain I is a small category.

- Define $L := \{\vec{x} \in \prod_{i \in I} Di | \forall u : i \to j \in I. x_j = Dux_i\}$ and $\ell_i : L \to Di$ to be the restriction of the i-th component projection. Show that $\langle L, \ell\rangle$ is a limiting cone for D.
- Let R to be the relation on the disjoint union $\coprod_{i \in I} Di$ given by $\langle i, x\rangle \, R \, \langle j, y\rangle$ when there is some $u : i \to j$ with $Dux = y$. Let $\equiv_R$ be the reflexive-transitive-symmetric closure of $\equiv_R$. Define a cocone by setting $C := \coprod_{i \in I} Di / \equiv_R$ and c_i mapping each $x \in Di$ to $[\langle i, x\rangle]$, the $\equiv_R$-equivalence class of $\langle i, x\rangle$. Show that $\langle C, c\rangle$ is a colimiting cocone for D. ◿

Let $\mathcal{D}$ be a class of diagrams in a category $\mathcal{C}$. We say that $\mathcal{C}$ is $\mathcal{D}$-*complete* when it has limit cones for all diagrams in $\mathcal{D}$, and $\mathcal{D}$-*cocomplete* when it has colimiting cocones for all diagrams in $\mathcal{D}$. By default, $\mathcal{D}$ is the class of all small diagrams. The category **Set** is therefore complete and cocomplete. We can often use this fact to transfer limits and colimits along functors into other categories.

Let $H : \mathcal{B} \to \mathcal{C}$ be a functor, and $D : I \to \mathcal{B}$ a diagram. If $c : \mathbf{\Delta} C \to D$ is a D-cone, then $Hc : \mathbf{\Delta} HC \to H \circ D$ is an $H \circ D$-cone. We say that H:

- preserves D-limits when, for every limiting cone $\langle L, \ell\rangle$, the cone $\langle HL, H\ell\rangle$ is limiting for $H \circ D$;
- reflects D-limits when, for every D-cone $\langle L, \ell\rangle$, if the cone $\langle HL, H\ell\rangle$ is $H \circ D$-limiting, then $\langle L, \ell\rangle$ is limiting (and then H preserves this limit); and
- lifts D-limits when, for every $H \circ D$-cone $\langle L', \ell'\rangle$ there is a D-limiting cone $\langle L, \ell\rangle$ and a cone isomorphism $\langle HL, H\ell\rangle \cong \langle L', \ell'\rangle$.

We extend these to a class of diagrams $\mathcal{D}$ by saying that H preserves/reflects/lifts $\mathcal{D}$-limits of the class if it does so for each diagram in $\mathcal{D}$. Finally, we say that:

- H is $\mathcal{D}$-continuous when it preserves $\mathcal{D}$-limits;
- H *generates* $\mathcal{D}$-limits when it preserves and lifts $\mathcal{D}$-limits; and
- H *creates* $\mathcal{D}$-limits when it preserves, reflects, and lifts $\mathcal{D}$-limits.

We define analogous concepts for colimits.

◸**3.27.** Show that $\llcorner - \lrcorner : \mathbf{Meas} \to \mathbf{Set}$ lifts limits, but does not reflect limits. ◿

◸**3.28.** Show that if $H : \mathcal{B} \to \mathcal{C}$ lifts $\mathcal{D}$-limits and $\mathcal{C}$ is $\mathcal{D}$-complete, then $\mathcal{B}$ is also $\mathcal{D}$-complete and H generates $\mathcal{D}$-limits. Deduce that **Meas** is complete. ◿

◸**3.29.** Show that the Yoneda embedding preserves limits. ◿

Let $\mathcal{C}$ be a locally small category. A functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is *representable* when there is some object X and a natural isomorphism $\rho : \mathbf{y}X \xrightarrow{\cong} F$. We call the object X the *representing object* and the isomorphism ρ the *representation*.

◸**3.30.** Let $H : \mathcal{B} \to \mathcal{C}$ be a functor between locally small categories. Show that a universal arrow $\langle X, f\rangle$ from A to H is just a representation $\rho : \mathbf{y}_{\mathcal{B}^{\mathrm{op}}} X \xrightarrow{\cong} \lambda x.\mathcal{C}(A, Hx)$, and the translation between f and ρ is given by the Yoneda lemma:

$$f \in \mathcal{C}(A, HX) \qquad \overset{\Upsilon_X}{\longleftrightarrow} \qquad \rho \in \widehat{\mathcal{B}^{\mathrm{op}}}(\mathbf{y}X, \lambda x.\mathcal{C}(A, Hx))$$ ◿

Solution. A representation ρ is a family of bijections, natural in Y, between two hom-sets:

$$\frac{A \longrightarrow HY}{X \longrightarrow Y} \tag{2}$$

The Yoneda lemma gives an arrow $f := \Upsilon\rho$. The naturality of ρ implies that for all $h : X \to Y$:

$$\begin{tikzcd}
\mathcal{B}(X,X) \arrow[r, "\rho_X"] \arrow[d, bend right, "\mathcal{B}(X,h)"'] \arrow[d, bend left, "\mathbf{y}_{\mathcal{B}^{\mathrm{op}}} h"] \arrow[d, phantom, "="] \arrow[dr, phantom, "="] & \mathcal{C}(A,HX) \arrow[d, "\mathcal{C}(A,Hh)"] \\
\mathcal{B}(X,Y) \arrow[r, "\rho_Y"'] & \mathcal{C}(A,HY)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{id}_X \arrow[r, mapsto, "\rho_X"] \arrow[d, mapsto, "\mathcal{B}(X,h)"'] \arrow[dr, phantom, "="] & \rho_X(\mathrm{id}_X) = \Upsilon_X\rho =: f \arrow[d, mapsto, "\mathcal{C}(A,Hh)"] \\
h \circ \mathrm{id}_X = h \arrow[r, mapsto, "\rho_Y"'] & \rho_Y h = Hh \circ f
\end{tikzcd}$$

So the bijection ρ_Y acts by $\lambda h.Hh \circ f$, and that's the universality of the arrow $\langle X, f\rangle$.

Conversely, a universal arrow $\langle X, f\rangle$ induces a bijective correspondences as in (2) given by $\rho_Y(h : X \to Y) := Hh \circ f = \mathcal{C}(A, Hh)f = (\Upsilon_X^{-1} f)_Y h$, and so $\rho = \Upsilon_X^{-1} f$, and also Y-natural. ◢

3.3 Adjunctions

The input to the universal arrow concept in the previous section is a functor and an object A of $\mathcal{C}$. By currying the object, we arrive at the concept of an adjoint:

- Let $U : \mathcal{B} \to \mathcal{C}$ be a functor. A *left adjoint to* U, $\langle FA \in \mathcal{B}, \eta_A : A \to U(FA)\rangle_{A \in \mathcal{C}}$, is an assignment, for each object $A \in \mathcal{C}$, of a universal arrow $\langle FA, \eta_A\rangle$ from A to U.
- Similarly, let $F : \mathcal{C} \to \mathcal{B}$ be a functor. A *right adjoint to* F, $\langle UX, \varepsilon_X : F(UX) \to X\rangle_{X \in \mathcal{B}}$, is an assignnment, for each object $X \in \mathcal{B}$, of a universal arrow $\langle UX, \varepsilon_X\rangle$.

So an adjoint is a simultaneous assignment of universal arrows.

So far we've seen plenty of examples of adjoints:

◸**3.31.** Show that the functors $\llcorner - \lrcorner : \mathbf{Meas} \to \mathbf{Set}$ and $\llcorner - \lrcorner : \mathbf{Geom} \to \mathbf{Set}$ have both a left and right adjoints. ◿

◸**3.32.** Show that the functor $\llcorner - \lrcorner_{\mathbf{Geom}} : \mathbf{Meas} \hookrightarrow \mathbf{Geom}$ has a right adjoint. ◿

◸**3.33.** Show that the functor $\llcorner - \lrcorner : \mathbf{Equiv}_A \hookrightarrow \mathbf{Rel}_A$ has a left adjoint. ◿

◸**3.34.** Show that the diagonal functor $\mathbf{\Delta} : \mathbf{Meas} \to \mathbf{Meas}^2$ has a right adjoint. ◿

◸**3.35.** Show that every diagonal functor $\mathbf{\Delta} : \mathbf{Set} \to \mathbf{Set}^I$ for a small category I, has both a left and a right adjoint. Every diagonal functor $\mathbf{\Delta} : \mathbf{Meas} \to \mathbf{Meas}^I$ for a small category I has a right adjoint. ◿

Let $U : \mathcal{B} \to \mathcal{C}$ be a functor with a right adjoint $\langle F, \eta \rangle$. By Ex.3.30, each universal arrow $\langle FA, \eta_A \rangle$ comes from a representation $\rho_A : \mathbf{y}(FA) \xrightarrow{\cong} \lambda x.\mathcal{C}(A, Ux)$, so we have a collection of bijections, indexed by both $A \in \mathcal{C}$ and $X \in Y$:

$$\rho_{A,X} : \mathcal{B}(FA, X) \xrightarrow{\cong} \mathcal{C}(A, UX) \qquad \rho_{A,X} : \left(FA \xrightarrow{h} X\right) \mapsto \left(A \xrightarrow{\eta_A} U(FA) \xrightarrow{Uh} UX\right)$$

It is natural in X, but if we want it to be natural in A, we need to equip F with a functorial action on morphisms.

◸**3.36.** Show that there is exactly one action on morphisms such that:

- $F : \mathcal{C} \to \mathcal{B}$ is a functor; and
- $\rho : (\lambda x, y.\mathcal{B}(Fx, y)) \xrightarrow{\cong} (\lambda x, y.\mathcal{C}(x, Uy))$ is a natural transformation (and so forms a natural isomorphism). ◿

An *adjunction* from $\mathcal{C}$ to $\mathcal{B}$ is a tuple $\langle F, G, \rho \rangle$ consisting of:

- Two functors $F : \mathcal{C} \to \mathcal{B}$, the *left adjoint* and $G : \mathcal{B} \to \mathcal{C}$, the *right adjoint*; and
- A natural isomorphism $\rho : (\lambda x, y.\mathcal{B}(Fx, y)) \xrightarrow{\cong} (\lambda x, y.\mathcal{C}(x, Uy))$ called the *mate bijection.*

By Ex.3.36, each adjoint extends to a unique adjunction. This process exhausts all adjunctions. Indeed, in an adjunction, each bijection $\rho_A : \mathbf{y}FA \xrightarrow{\cong} \lambda y.\mathcal{C}(A, y)$ is a representation. By Ex.3.30, setting $\eta_A := \Upsilon_{FA}\rho_A : A \to UFA$ gives a simultaneous assignment $\langle FA, \eta_A \rangle_{A \in \mathcal{C}}$ of universal arrows from A to U, i.e., a left adjoint to U. Using a similar argument for right adjoints, we have that an adjunction is just an adjoint (left or right) to the appropriate functor in the adjunction.

Given an adjunction, as we've seen, the universal arrows are given by:

$$\eta_A := \rho_{A,FA}\mathrm{id}_{FA} = \Upsilon_{FA}(\rho_A) \qquad \varepsilon_X := \rho^{-1}_{UX,X}\mathrm{id}_{UX} = \Upsilon_{UX}(\rho^{-1}_X)$$

◸**3.37.** Show that $\eta : \mathrm{Id}_{\mathcal{C}} \to U \circ F$ and $\varepsilon : F \circ U \to \mathrm{Id}_{\mathcal{B}}$ are natural. Can you do it by appealing to the naturality of the Yoneda lemma? ◿

Solution. We'll do so for η, the proof for ε is similar. We need to show, for $f : A \to B$ in $\mathcal{C}$:

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & UFA \arrow[d, "UFf"] \\
B \arrow[r, "\eta_B"'] & UFB
\end{tikzcd}$$

The Yoneda lemma in question is the natural isomorphism:

$$\Upsilon : \big(\lambda x.\hat{\mathcal{B}}(\mathbf{y}x, \lambda y.\mathcal{C}(A, Uy))\big) \xrightarrow{\cong} (\lambda x.\mathcal{C}(A, Ux))$$

$$\begin{tikzcd}
\rho_A \arrow[rrr, mapsto] \arrow[ddd, bend right] & & & \eta_A \arrow[ddd, bend left] \\
& \hat{\mathcal{B}}(\mathbf{y}FA, \lambda y.\mathcal{C}(A, Uy)) \arrow[r, "\Upsilon_{FA}"] \arrow[d, "{\hat{\mathcal{B}}(\mathbf{y}(Ff), \lambda y.\mathcal{C}(A, Uy))}"'] \arrow[dr, phantom, "="] & C(A, UFA) \arrow[d, "{\mathcal{C}(A, UFf)}"] & \\
& \hat{\mathcal{B}}(\mathbf{y}FB, \lambda y.\mathcal{C}(A, Uy)) \arrow[r, "\Upsilon_{FB}"'] & C(A, UFB) & \\
(\lambda h.\rho_A(h \circ Ff)) \underset{\substack{\uparrow \\ \rho\text{-naturality}}}{=} (\lambda h.\rho_B h \circ f) \arrow[rrr, mapsto] & & & \eta_B \circ f = UFf \circ \eta_A
\end{tikzcd}$$

as we wanted. ◢

We have $\varepsilon_{FA} \circ F\eta_A = \rho^{-1}_{FA}(\eta_A) = \mathrm{id}_{FA}$ and similarly $U\varepsilon_X \circ \eta_{UX} = \mathrm{id}_{UX}$, and we arrived at the following concept.

A *formal adjunction* $\langle F, G, \eta, \varepsilon\rangle$ consists of:

- Two functors $F : \mathcal{C} \to \mathcal{B}$, the *left adjoint* and $G : \mathcal{B} \to \mathcal{C}$, the *right adjoint*; and
- Two natural transformations $\eta : \mathrm{Id}_{\mathcal{C}} \to U \circ F$, the *unit*, and $\varepsilon : F \circ U \to \mathrm{Id}_{\mathcal{B}}$, the *counit*.

satisfying the following *triangle equalities*, for every $X \in \mathcal{B}$ and $A \in \mathcal{C}$:

$$\begin{tikzcd}
FA \arrow[rr, equal] \arrow[dr, "F\eta_A"'] & {} \arrow[d, phantom, "="] & FA \\
& FUFA \arrow[ur, "\varepsilon_{FA}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& UFUX \arrow[dr, "U\varepsilon_X"] \arrow[d, phantom, "="] & \\
UX \arrow[ur, "\eta_{UX}"] \arrow[rr, equal] & {} & UX
\end{tikzcd}$$

The term ‘formal’ here is used in the Australian sense: it involves categories, functors (morphisms between categories), and natural transformations (morphisms between functors), and so can be generalised to ‘formal’ categories that have 0-cells (objects), 1-cells (morphisms between 0-cells), and 2-cells (morphisms between 1-cells).

We’ve shown that every adjunction gives rise to a formal adjunction. The mate isomorphism ρ is determined by η as:

$$\rho_{A,X} h \coloneqq Hh \circ \eta_A = (\Upsilon^{-1}_{FA}\eta_A)_X h \tag{3}$$

Therefore, this formal adjunction is determined uniquely. To see this process is exhaustive, take any formal adjunction, and set $\rho_A \coloneqq \Upsilon^{-1}_{FA}\eta_A : \mathbf{y}FA \to \lambda y.C(A, Uy)$ using the Yoneda lemma as in (3). Then ρ_A is natural.

◸**3.38.** Show that ρ_A is an isomorphism. ◿

Solution. Define $\rho^{-1}_{A,X} h' \coloneqq \varepsilon_X \circ Fh'$ and show it is inverse to $\rho_{A,X}$ by direct calculation. For example, take any $h' : A \to UX$ in $\mathcal{C}$, and show that $h = \rho_{A,X}(\rho^{-1}_{A,X} h) = U\varepsilon_X \circ UFh' \circ \eta$:

$$\begin{tikzcd}
A \arrow[rr, "\eta_A"] \arrow[dr, "h'"] \arrow[dd, "h'"'] & & UFA \arrow[dd, "UFh'"] \arrow[dl, phantom, "\overset{\eta\text{-nat}}{=}"] \\
\arrow[r, phantom, "\overset{\text{id}}{=}"] & UX \arrow[dl, equal] \arrow[dr, "\eta_{UX}"] \arrow[d, phantom, "\overset{\text{triangle}}{=}"] & \\
UX & \, & UFUX \arrow[ll, "U\varepsilon"]
\end{tikzcd}$$

as we wanted. ◢

Since ρ_A is a natural isomorphism, by Ex.3.30, we have a universal arrow $\langle FA, \eta_A \rangle$ from A to U, so we have a left adjoint to U, hence an adjunction.

◸**3.39.** Check that the resulting formal adjunction is our given formal adjunction. ◿

Summarising, a formal adjunction is just an adjunction, which in turn is both just a left adjoint F to a functor $U : \mathcal{B} \to \mathcal{C}$ and just a right adjoint U to a functor $F : \mathcal{C} \to \mathcal{B}$. We write $\rho, \langle \eta, \varepsilon \rangle : F \dashv U : \mathcal{B} \to \mathcal{C}$ where ρ is the mate isomorphism of the adjunction, and η is the unit and ε the counit of the formal adjunction.

◸**3.40.** Let $\langle \eta, \varepsilon \rangle : F \dashv U : \mathcal{B} \to \mathcal{C}$ and let $H : \mathcal{C} \to \mathcal{D}$. Show that if $\langle A, v \rangle$ is a universal arrow from V to H, then $\left\langle FA, V \xrightarrow{v} HX \xrightarrow{H\eta_A} HU(FX) \right\rangle$ is a universal arrow from V to $H \circ U$.

Deduce that given two composable adjunctions $F_1 \dashv U_1$ and $F_2 \dashv U_2$:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "F_2"] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, bend left, "U_2"] \arrow[r, bend left, "F_1"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "U_1"]
\end{tikzcd}$$

their composition is also an adjunction $F_1 \circ F_2 \dashv U_2 \circ U_1$. ◿

◸**3.41.** Show that right adjoints preserve limits and left adjoints preserve colimits. ◿

References