## 6 Quasi-Borel spaces

Practice the basic definitions of qbses and their morphisms.

◸**6.1.** We can equip the real numbers with the structure of a qbs:

- The points are the real numbers.
- The random elements are the Borel measurable functions $\alpha : \mathbb{R} \to \mathbb{R}$

We'll write more succinctly below: $\mathbb{R} := \langle \llcorner \mathbb{R} \lrcorner, \mathbf{Meas}(\mathbb{R}, \mathbb{R}) \rangle$.

- Check that $\mathbb{R}$ satisfies the qbs axioms.
- Show that a function $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable iff it is a qbs morphism. ⊿

◸**6.2.** Let $X$ be a set. The *indiscrete qbs over* $X$ has all functions as random elements:

$$\llcorner X \lrcorner_{\mathbf{Qbs}} := \langle X, \mathbf{Set}(\mathbb{R}, X) \rangle$$

- Check that $\llcorner X \lrcorner_{\mathbf{Qbs}}$ satisfies the qbs axioms.
- Let $A$ be any qbs. Show that every function $f : \llcorner A \lrcorner_{\mathbf{Set}} \to X$ is a qbs morphism:

$$f : A \to \llcorner X \lrcorner_{\mathbf{Qbs}}$$

⊿

◸**6.3.** A *qbs structure* on a set $X$ is a collection $\mathcal{R} \subseteq X^{\mathbb{R}}$ of functions closed under the qbs axioms. A function $\alpha : \mathbb{R} \to X$ is $\sigma$*-simple* when:

- The image $\alpha[\mathbb{R}]$ is countable; and
- For every $x \in \alpha[\mathbb{R}]$, the preimage $\alpha^{-1}[x] \subseteq \mathbb{R}$ is a Borel set.

Show the $\sigma$-simple functions are the smallest (w.r.t. set inclusion) qbs structure on $X$. ⊿

◸**6.4.** Let $A, B, C$ be qbses. Show that the following functions are qbs morphisms:

- Constant functions: for every $b \in \llcorner B \lrcorner_{\mathbf{Set}}$, the function $(\lambda a.b) : A \to B$.
- Identity functions: $\mathrm{id} := (\lambda a.a) : A \to A$.
- If $f : B \to C$ and $g : A \to B$ are qbs morphisms then so is the composition $f \circ g : A \to C$.
- Every $\sigma$-simple functions $\alpha : \mathbb{R} \to A$. ⊿

◸**6.5.** Let $X$ be a set. The *discrete qbs over* $X$ has the $\sigma$-simple functions as random elements:

$$\ulcorner X \urcorner^{\mathbf{Qbs}} := \langle X, \{\alpha : \mathbb{R} \to X | \alpha \text{ is } \sigma\text{-simple}\} \rangle$$

By Ex.6.3, it is a qbs. Let $A$ be any qbs. Show that every function $f : X \to \llcorner A \lrcorner_{\mathbf{Set}}$ is a qbs morphism:

$$f : \ulcorner X \urcorner^{\mathbf{Qbs}} \to A$$

⊿

◸**6.6.** Let $A, B$ be isomorphic qbses. Show that their sets of points and their sets of random elements are in bijection:

$$\llcorner A \lrcorner \cong \llcorner B \lrcorner \qquad \mathcal{R}_A \cong \mathcal{R}_B$$

(Recall from Ex.2.7 that two spaces $A, B$ are isormorphic when there are two morphisms $f : A \to B$ and $g : B \to A$ that are each other's inverses: $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$.) ◿

◸**6.7.** Show that the three spaces:

- $\mathbb{R}$, defined in Ex.6.1;
- $\llcorner \mathbb{R} \lrcorner_{\mathbf{Qbs}}$, defined in Ex.6.2; and
- $\ulcorner \mathbb{R} \urcorner^{\mathbf{Qbs}}$, defined in Ex.6.5

are pairwise non-isomorphic qbses. ◿

◸**6.8.** Let $f : A \to B$ be a qbs morphism. Show:

- $f$ is surjective iff $f$ is an epimorphism in **Qbs**.
- $f$ is injective iff $f$ is a monomorphism in **Qbs**. ◿

◸**6.9.** We have a functor $\llcorner - \lrcorner_{\mathbf{Set}} : \mathbf{Qbs} \to \mathbf{Set}$ sending each qbs $A$ to its set of points.

- Define the action on morphisms, and show it is functorial and faithful.

Show:

- $\llcorner - \lrcorner_{\mathbf{Set}} : \mathbf{Qbs} \to \mathbf{Set}$ has both a left and a right adjoint. What are the unit, counit, and mate representations of each adjunction?
- The functor $\llcorner - \lrcorner_{\mathbf{Set}}$ is *essentially surjective*: every set is isomorphic to a set of points of some space.
- These left and right adjoints are fully-faithful, and neither essentially surjective. ◿

## References