5 Sequences

◸**5.1.** Show that the following sets are Borel in the extended real numbers $[-\infty, \infty]$:

- The set of converging sequences (including sequences whose limit $\pm\infty$):

$$\mathrm{Converge}[-\infty,\infty] := \left\{\vec{r} \in [-\infty,\infty]^{\mathbb{N}} \middle| \exists \lim_{n\to\infty} r_n\right\}$$

- For every $a \in [-\infty, \infty]$, the set of sequences that converge to a:

$$\mathrm{ConvergeTo}\, a := \left\{\vec{r} \in [-\infty,\infty]^{\mathbb{N}} \middle| \lim_{n\to\infty} r_n = a\right\}$$

- The set of convergence rates:

$$\mathrm{ConvergenceRate} := \left\{\vec{r} \in (0,\infty] \middle| \lim_{n\to\infty} r_n = 0\right\}$$ ◿

◸**5.2.** Show that the following higher-order operations are measurable:

- $\lim : \mathrm{Converge}[\infty,\infty] \to [-\infty,\infty]$
- $\liminf, \limsup : [-\infty,\infty]^{\mathbb{N}} \to [-\infty,\infty]$
- $\arg\min, \arg\max : [-\infty,\infty]^{\mathbb{N}} \to \mathbb{N}_\perp$
- $\min : \mathcal{B}_{\mathbb{N}} \setminus \{\varnothing\} \to \mathbb{N}$, where $\mathcal{B}_{\mathbb{N}}$ is the measurable space structure induced by identifying the measurable subsets of $\mathbb{N}$ with their indicator functions in the countable-product measurable space $2^{\mathbb{N}}$. ◿

◸**5.3.** For every measurable space X, we may adjoin a new element $\perp$ called ‘bottom’ representing the undefined value, and making the singleton $\{\perp\}$ measurable. Explicitly:

- The points are the disjoint union of the points in X and $\perp$: $\llcorner X_\perp \lrcorner := \{\perp\} \amalg \llcorner X \lrcorner$.
- The measurable sets are generated by those of X and $\{\perp\}$:

$$\mathcal{B}_{X_\perp} := \sigma(\{\{\iota_1\perp\}\} \cup \iota_2[[\mathcal{B}_X]])$$

We can use the undefined value to define partial measurable functions. Show that the following higher-order operations are measurable:

- $\lim : [-\infty,\infty]^{\mathbb{N}} \to [-\infty,\infty]_\perp$
- $\inf, \sup : ([-\infty,\infty]_\perp)^{\mathbb{N}} \to [-\infty,\infty]$
- $\mathrm{compress} : (X_\perp)^{\mathbb{N}} \to \left(X^{\mathbb{N}}\right)_\perp$ for any measurable space X which compresses the sequence by removing any intermediate undefined values. ◿

◸**5.4.** Define a measurable function $\mathrm{approx}\_ : \mathrm{ConvergenceRate} \times \mathbb{R} \to \mathbb{Q}^{\mathbb{N}}$, such that each $\mathrm{approx}_{\vec{b}} r$ is a sequence $\vec{q}$ of rational numbers that converges to r at rate $\vec{b}$, so for all $n \in \mathbb{N}$: $|q_n - r| < b_n$. ◿

References