Quasi-Borel space exercises

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Abstract

Exercises on quasi-Borel spaces. I've prepared them to accompany my Marseille talk and the following Scottish Programming Languages and Verification (SPLV) Summer School about higher-order measure theory with quasi-Borel spaces.

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About these exercises. Some exercises are there to fill gaps in the mathematical development. Others can help you practice juggling the concepts involved. I also include exercises for exploring independently a part of the theory that's not directly required by the talk, but you might find interesting. The Marseille and SPLV meetings bring together practitioners and theoreticians with a mix of backgrounds. So I also include exercises that might help give you enough for the purpose of this course. For those exercises I also point what topics you should read further if you find this kind material fun. The exercises in Sec. 1–Sec. 5 are such, and are often covered by most introductory courses in their respective areas. So you should be able to tell at a glance whether you can skip them without losing the thread. In that case, you might want to help others who are less familiar with this material.

Getting help and reporting mistakes. Please never hesitate to get in touch. You can reach me directly by email. You may prefer to ask a question on the #qbs channel on the SPLS Zulip server: `spls.zulipchat.com`. Unfortunately, these exercises undoubtly contain mistakes. If you've found one, or if you're stuck, please reach out!

1 Borel sets basics

Try these exercises if you're new to Borel sets of real numbers.

◸**1.1.** Show that the Borel sets are closed under:

- finite unions;
- countable intersections;
- translations:

$$A \in \mathcal{B}_{\mathbb{R}} \qquad \Longrightarrow \qquad r + [A] \coloneqq \{r + a | a \in A\} \in \mathcal{B}_{\mathbb{R}}$$ ◿

◸**1.2.** Show that the following sets are Borel ($a, b \in \mathbb{R}$):

- $[a, b]$;
- $\{a\}$;
- $(-\infty, a]$;
- $[a, b)$;
- $\mathbb{Q}$: the rational numbers ◿

Recall the *limit superior* and *limit inferior* operations on sequences of subsets $\vec{A} \subseteq X^{\mathbb{N}}$, thinking of them as subsets that vary in discrete time:

$\boldsymbol{\limsup_{n\to\infty} A_n \coloneqq \bigcap_{k\in\mathbb{N}} \bigcup_{\ell\geq k} A_\ell}$**:** elements appearing *infinitely often* in the sequence;
$\boldsymbol{\liminf_{n\to\infty} A_n \coloneqq \bigcup_{k\in\mathbb{N}} \bigcap_{\ell\geq k} A_\ell}$**:** elements appearing in *almost all* the sequence;
$\boldsymbol{\lim_{n\to\infty} A_n \coloneqq \liminf_n A_n = \limsup_n A_n}$ when the two limits coincide.

If the elements of the sequence are Borel, so are the two limits.
For example, use sequences 3-valued indexed by natural numbers $\vec{x} \in \{0, 1, \mathrm{wait}\}^{\mathbb{N}}$ to represent possibly-blocking streams of bits. Let $A_n \coloneqq \{\vec{x} | x_n \neq \mathrm{wait}\}$. Then:

- $\limsup_n A_n$ are the streams that always produce more output; while
- $\liminf_n A_n$ are the streams that eventually stop blocking.

◸**1.3.** Practice manipulating limits of sets.

- (Taken from Wikipedia.) Calculate the two limits for the following sequences:
  - $\left\langle \left(-\frac{1}{n}, 1 - \frac{1}{n}\right) \right\rangle_n$
  - $\left\langle \left(\frac{(-1)^n}{n}, 1 - \frac{(-1)^n}{n}\right) \right\rangle_n$
  - $\left\langle \left\{ \frac{i}{n} \middle| i = 0, \ldots, n \right\} \right\rangle_n$
- Show that:

$$\bigcap \vec{A} \subseteq \liminf \vec{A} \subseteq \limsup \vec{A} \subseteq \bigcup \vec{A}$$

- What happens to the two limits when $A_n \subseteq A_{n+1}$ and when $A_n \supseteq A_{n+1}$?
- This is the *indicator* function of a set $A \subseteq X$:

$$[- \in A] : X \to \{0, 1\}$$

$$[x \in A] \coloneqq \begin{cases} x \in A : & 1 \\ x \notin A : & 0 \end{cases}$$

  Show that:

- $\bigcup \vec{A} = \{x \in X | \sup_n [x \in A_n] = 1\}$
- $\limsup \vec{A} = \{x \in X | \limsup_n [x \in A_n] = 1\}$
- $\liminf \vec{A} = \{x \in X | \liminf_n [x \in A_n] = 1\}$
- $\bigcap \vec{A} = \{x \in X | \inf_n [x \in A_n] = 1\}$ ◿

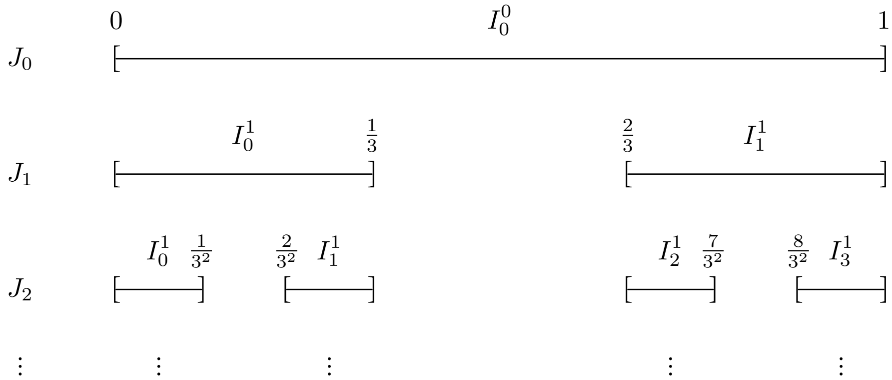
◸**1.4.** Let's construct the *Cantor set*. For each $n \in \mathbb{N}$, let $\mathbf{Fin}\, n \coloneqq \{0, \dots, n-1\}$ be the n-th cardinal. We define:

$$I : \coprod_{n=0}^{\infty} \mathbf{Fin}\, 2^n \to \left\{[a,b] \middle| b - a = \tfrac{1}{3^n}\right\} \subseteq \mathcal{B}_{\mathbb{R}}$$

as follows, writing $I^n_k \coloneqq I(\iota_n k)$ for each $n \in \mathbb{N}$ and $k \in \mathbf{Fin}\, 2^n$:

$$I^0_0 \coloneqq [0,1] \qquad I^{n+1}_{2k} \coloneqq [\min I^{n+1}_k, \tfrac{1}{3^{n+1}} + \min I^{n+1}_k] \qquad I^{n+1}_{2k+1} \coloneqq [\max I^{n+1}_k - \tfrac{1}{3^{n+1}}, \max I^{n+1}_k]$$

Each union $J_n \coloneqq \bigcup_{k \in \mathbf{Fin}\, 2^n} I^n_k$ drops the middle thirds in the preceding interval sequence:



Later we'll define the *Lebesgue* measure as the unique σ-additive function $\boldsymbol{\lambda} : \mathcal{B}_{\mathbb{R}} \to [0, \infty]$ that assigns to each interval its length.

- Show that $\langle \boldsymbol{\lambda} J_n \rangle_n$ *vanishes*: $\lim_{n \to \infty} \boldsymbol{\lambda} J_n = 0$, by calculating each number in the sequence.
- The *Cantor set* is the limit $\mathbb{G} \coloneqq \lim_n J_n$. Show that $\boldsymbol{\lambda} \mathbb{G} = 0$.
- Find a bijection $\mathbb{G} \cong \mathbb{T} \coloneqq \mathbb{2}^{\mathbb{N}}$ where $\mathbb{2} \coloneqq \mathbf{Fin}\, 2$.
- If you know some topology, equip $\mathbb{G} \hookrightarrow \mathbb{R}$ with the sub-space topology w.r.t. the open subsets of $\mathbb{R}$ and $\mathbb{T} = \prod_{n \in \mathbb{N}} \mathbb{2}$ with the product topology w.r.t. the discrete topology on $\mathbb{2}$. Find a homeomorphism $\mathbb{G} \cong \mathbb{T}$. ◿

2 Measurable spaces and functions

Try these exercises if you're new to measure theory and are curious about it.

◸**2.1.** Show that each subset is Borel in each measurable space:

- The diagonal $\left\{\langle r, r\rangle \in \mathbb{R}^2 \middle| r \in \mathbb{R}\right\}$ in the Euclidean plane $\mathbb{R}^2$.
- The 3-dimensional open ball $\left\{\langle x, y, z\rangle \in \mathbb{R}^3 \middle| x^2 + y^2 + z^2 < 1\right\}$ in the Euclidean space $\mathbb{R}^3$.
- The 2-dimensional sphere $\left\{\langle x, y, z\rangle \in \mathbb{R}^3 \middle| x^2 + y^2 + z^2 = 1\right\}$ in the Euclidean space $\mathbb{R}^3$.

If you're unsure how to approach the exercise, try the rest of this section first. ◿

◸**2.2.** Prove that the following functions over $\mathbb{R}$ are measurable, for all $r \in \mathbb{R}$:

- $(r+) \coloneqq \lambda s.r + s$
- $(r\cdot) \coloneqq \lambda s.r \cdot s$ ◿

We can organise measurable spaces and functions into a category called **Meas**: the measurable spaces are the objects and the measurable functions are the morphisms between these objects. You already know another category: **Set**. Its objects are sets and its morphisms are functions between those sets. This course isn't about category theory, but we will take advantage of category theory to help us relate concepts that live in different areas of mathematics. So if you never worked with categories before, you can use this course to learn a bit more about categories. In that case, please take full advantage of myself and your categorically-savvy course-mates!
If you are such a categorically-savvy person, you already covered the next few exercises in the past and may want to skip to Ex.2.8.

◸**2.3.** Let's spell out the category structure of **Meas**:

- Objects are measurable spaces X, Y;
- Morphisms $f : X \to Y$ are measurable functions of the same type.
- Identities $\mathrm{id}_X : X \to X$ are the identity functions $\lambda x.x$ of the same type.
- The composition of $f : Y \to Z$ and $g : X \to Y$ is the composed function $f \circ g : X \to Z$.

Show the implicit statements in the last two clauses:

- The identity function is a measurable functions $\mathrm{id}_X : X \to X$.
- The composition is a measurable function $f \circ g : X \to Z$. ◿

Having spelled out the structure, we should now check that this structure is a category:

◸**2.4.** Show that:

- identities are neutral w.r.t. composition — for all $f : X \to Y$: $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$; and
- composition is associative: $f \circ (g \circ h) = (f \circ g) \circ h$. ◿

You may have found 1-line proofs for each of the category axioms for **Meas**. This may feels silly and tedious. It also usually means there's a structural reason why those proofs work. Here is one. There is a functor $\llcorner - \lrcorner : \mathbf{Meas} \to \mathbf{Set}$, that is, there is an assignment:

- to each measurable space X, we assign set its set of points $\llcorner X \lrcorner$;
- to each measurable function $f : X \to Y$, we assign its corresponding function between the corresponding sets of points $f : \llcorner X \lrcorner \to \llcorner Y \lrcorner$;

and this assignment respects identities and composition:

◸**2.5.** Show that:

- $\llcorner \mathrm{id}_X \lrcorner = \mathrm{id}_{\llcorner X \lrcorner}$ for every measurable space X; and
- $\llcorner f \circ g \lrcorner = \llcorner f \lrcorner \circ \llcorner g \lrcorner$ for every pair of composable measurable functions $X \xrightarrow{g} Y \xrightarrow{f} Z$. ⊿

Equations between functions (and more generally, morphisms) leave the intermediate spaces implicit in the background. We can mention both the spaces and the last two equations diagrammatically:

$$
\begin{tikzcd}
X \arrow[d, bend right, "\llcorner \mathrm{id}_X \lrcorner"'] \arrow[d, equal, bend left, "\mathrm{id}_{\llcorner X \lrcorner}"] \arrow[d, phantom, "="] \\
X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rr, "\llcorner f \lrcorner"] \arrow[d, "\llcorner g \circ f \lrcorner"'] & = & Y \arrow[dll, "\llcorner g \lrcorner"] \\
Z & &
\end{tikzcd}
$$

Vertices in the diagrams are objects, and directed edges are labelled by morphisms between these objects. We'll use a stretched equality notation to mark edges labelled by identity morphisms, and often omit the actual label. Each face has a source and a sink, and two paths from the source to the sink comprising of composable morphisms. The equality sign on a face states an equality between the composion of the morphisms on the two paths around the face. In the left diagram, it means $\llcorner \mathrm{id}_X \lrcorner = \mathrm{id}_{\llcorner X \lrcorner}$ and on the right diagram, it means $\llcorner f \circ g \lrcorner = \llcorner f \lrcorner \circ \llcorner g \lrcorner$.

Let $\mathcal{B}$ and $\mathcal{C}$ be category *structures*, so they have objects, morphisms, identities, and composition operators, but we make no assumptions that identities are neutral or composition is associative. A functor $F : \mathcal{B} \to \mathcal{C}$ is *faithful* when, for every pair of morphisms of the same type $f, g : X \to Y$ in $\mathcal{B}$, we have: $Ff = Fg \implies f = g$. So the functorial action on morphisms is injective.

◸**2.6.** Prove:

- The functor $\llcorner - \lrcorner : \mathbf{Meas} \to \mathbf{Set}$ is faithful.
- Faithful functors *reflect* categories: if $F : \mathcal{B} \to \mathcal{C}$ is faithful and $\mathcal{C}$ is a category, then $\mathcal{B}$ is also a category.
- Deduce that **Meas** is a category. ⊿

This kind of 'short-cut' is not a short-cut at all: we replaced 3×1-line proofs with the same 3×1-line proofs, merely done abstractly, *and* had to prove $\llcorner - \lrcorner$ is a functor, which involves 2 additional proofs.

My answer, and it may not be *your* answer, is that being able to relate concepts in **Meas** and **Set** and how to transfer properties (like being a category) across these relationships is a useful technique, and it's worth learning. Here are a few more simple examples:

◸**2.7.** A morphism $f : X \to Y$ in a category $\mathcal{C}$ is an *isomorphism* when there is a morphism $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$:

$$
\begin{tikzcd}
X \arrow[rr, equal] \arrow[dr, "f"'] & = & X \arrow[dr, "f"] & \\
& Y \arrow[ur, "g"] \arrow[rr, equal] & = & Y
\end{tikzcd}
$$

Show the following:

- Every functor $F : \mathcal{B} \to \mathcal{C}$ preserves isomorphisms: if $f : X \to Y$ is an isomorphism in $\mathcal{B}$, then $Ff : FX \to FY$ is an isomorphism in $\mathcal{C}$.
- Faithful functors reflect isomorphism *pairs*: for all $X \xrightarrow{f} Y \xrightarrow{g} X$ in $\mathcal{B}$, if Ff and Fg are each others' inverses in $\mathcal{C}$ then f and g are each others' inverses in $\mathcal{B}$.
- The faithful functor $\llcorner - \lrcorner$: **Meas** $\to$ **Set** does not reflect isomorphisms: there is a measurable function $f : X \to Y$ that is not an isomorphism, but its underlying function $f : \llcorner X \lrcorner \to \llcorner Y \lrcorner$ is bijective. ◿

Like any formalism, categories takes practice to pick the vocabulary up and to use it effectively, for example, only when it's needed. In the rest of the course, every statement involving categories will be accompanied by its non-categorical formulation, or may be safely skipped. Whether or not you choose to use the language of categories is up to you. At the very least, these statements offer another source of exercise for you.

◸**2.8.** Let V be a measurable space and $A \subseteq \llcorner V \lrcorner$ any subset.

- Prove that $\mathcal{B}_A \coloneqq \{U \cap A | U \in \mathcal{B}_V\}$ is a σ-algebra.
- Prove that if A is measurable, i.e., $A \in \mathcal{B}_V$, then $\mathcal{B}_A = \{U \in \mathcal{B}_V | U \subseteq A\}$.
- Show that the inclusion is a measurable function:

$$\begin{aligned} &i : A \subseteq V \\ &ix \coloneqq x \end{aligned}$$

- Let $f : V \to W$ be a measurable function then the restriction of f to A is measurable:

$$\begin{aligned} &f|_A : A \to W \\ &f|_A\, x \coloneqq fx \end{aligned}$$

◿

◸**2.9.** Prove that the following functions are measurable:

- $(\frac{1}{-}) : \mathbb{R}_{\neq 0} \to \mathbb{R}_{\neq 0}$ where $\mathbb{R}_{\neq 0} \coloneqq \mathbb{R} \setminus \{0\}$.
- $|-| : \mathbb{R} \to \mathbb{R}_{\geq 0}$ where $\mathbb{R}_{\geq 0} \coloneqq [0, \infty)$. ◿

◸**2.10.** Show that the inclusion $i : A \subseteq V$ is *cartesian* in the following way: for every measurable space W and measurable function $f : W \to V$ such that $\mathrm{Im}\,(f) \subseteq A$ there is a unique measurable function $h : W \to A$ with $f = i \circ h$:

$$\begin{tikzcd} \llcorner W \lrcorner \arrow[dr, "u"'] \arrow[drr, bend left, "\llcorner f \lrcorner"] & \arrow[d, phantom, "="] & \\ & \llcorner A \lrcorner \arrow[r, "\llcorner i \lrcorner"'] & \llcorner V \lrcorner \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} W \arrow[dr, dashed, "h"'] \arrow[drr, bend left, "f"] & \arrow[d, phantom, "="] & \\ & A \arrow[r, "i"'] & V \end{tikzcd}$$

in **Set** in **Meas** ◿

◸**2.11.** If you enjoyed the previous exercise, try generalising it. Let V be a measurable space, A a set, and $v : A \to \llcorner V \lrcorner$ a function. Show that v has a *cartesian lifting*:

- a measurable space $v^*V \in$ **Meas**, together with

- a measurable function $\dot{v} : v^*V \to V$,

such that:

- $\llcorner v^*V \lrcorner = A$ and $\llcorner \dot{v} \lrcorner = v$; and
- for every measurable function $f : W \to V$ and function $u : \llcorner W \lrcorner \to \llcorner A \lrcorner$, if the equation on the left holds then there is a unique measurable function $h : W \to v^*A$ satisfying $\llcorner h \lrcorner = u$ and the eqution on the right:

$$
\begin{tikzcd}
\llcorner W \lrcorner \arrow[dr, "u"'] \arrow[drr, bend left, "\llcorner f \lrcorner"] \arrow[drr, phantom, "="] & & \\
& A \arrow[r, "v"'] & \llcorner V \lrcorner
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
W \arrow[dr, dashed, "h"'] \arrow[drr, bend left, "f"] \arrow[drr, phantom, "="] & & \\
& v^*V \arrow[r, "\dot{v}"'] & V
\end{tikzcd}
$$

in **Set** in **Meas**

This fact states that the functor $\llcorner - \lrcorner : \mathbf{Meas} \to \mathbf{Set}$ is a Grothendieck fibration. We won't use this fact directly in the sequel. ◿

◸**2.12.** Let X be a set.

- Prove that every intersection of σ-algebras over X is a σ-algebra over X.

Let $\mathcal{U} \subseteq \wp A$ be a family of subsets of A. The σ-algebra $\sigma(\mathcal{U})$ *generated* by $\mathcal{U}$ is the smallest σ-algebra containing $\mathcal{U}$:

$$\sigma(\mathcal{U}) := \bigcap \{\mathcal{B} \subseteq \wp X | \mathcal{B} \text{ is a } \sigma\text{-algebra and } \mathcal{U} \subseteq \mathcal{B}\}$$

Prove:

- If $\mathcal{U} \subseteq \mathcal{V}$ then $\sigma(\mathcal{U}) \subseteq \sigma(\mathcal{V})$.
- If $\mathcal{U}$ is already a σ-algebra, then $\sigma(\mathcal{U}) = \mathcal{U}$.
- Let V be a measurable space and $f : \llcorner V \lrcorner \to X$ any function. Prove that $f : V \to \langle X, \sigma(\mathcal{U})\rangle$ is measurable iff for every $A \in \mathcal{U}$, we have $f^{-1}[U] \in \mathcal{B}_V$. ◿

◸**2.13.** Let X be a set, and set $\{[X]\} := \left\{U \subseteq X \middle| U \text{ is countable or } U^\complement \text{ is countable}\right\} \subseteq \wp X$.

- Show that $\{[X]\}$ is a σ-algebra over X. This σ-algebra is known as the countable-co-countable σ-algebra.
- Show that $\{[X]\} = \sigma(\{\{x\} | x \in X\})$ is the σ-algebra generated by the singletons. ◿

If you know some transfinite induction, you might want a predicative definition of $\sigma(\mathcal{U})$. In that case, have a look at the (extensive!) bunch of exercises in Sec. A.

◸**2.14.** Let $A \subseteq \llcorner V \lrcorner$ be a subset of a measurable space V. Show that if $\mathcal{U}$ generates the σ-algebra of V, then $\mathcal{U}' := A \cap [\mathcal{U}]$ generates the σ-algebra of the subspace A. ◿

◸**2.15.** Given families of subsets $\mathcal{U} \subseteq \wp X$ and $\mathcal{V} \subseteq \wp Y$, define their *box* σ-algebra:

$$\mathcal{U} \otimes \mathcal{V} := \sigma \{A \times B | A \in \mathcal{U}, B \in \mathcal{V}\}$$

Let U, V be two measurable spaces.

- Set $U \times V := \left\langle \llcorner U \lrcorner \times \llcorner V \lrcorner, \mathcal{B}_U \otimes \mathcal{B}_V \right\rangle$, and show that the cartesian projections $\pi_1 : U \times V \to U$ and $\pi_2 : U \times V \to V$ are measurable.
- Show that $\langle U \times V, \pi_1, \pi_2 \rangle$ is the categorical product: for every measurable space W and pair of measurable functions $f : W \to U$ and $g : W \to V$, there is a unique measurable function $\langle f, g \rangle : W \to U \times V$ such that:

$$\begin{tikzcd} & W \arrow[dl, bend right, "f"'] \arrow[d, dashed, "\langle f, g\rangle"] \arrow[dr, bend left, "g"] & \\ U & U \times V \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & V \end{tikzcd}$$

- Show that if $\mathcal{U}$ generates $\mathcal{B}_U$ and $\mathcal{V}$ generates $\mathcal{B}_V$, then $\mathcal{U} \otimes \mathcal{V} = \mathcal{B}_{U \times V}$. ⊿

◸**2.16.** Let $\vec{V} = \langle V_i \rangle_{i \in I}$ be an I-indexed family of measurable spaces.

- Find their categorical product $\prod_{i \in I} V_i$.
- Find an example family and an I-indexed family of measurable subsets $A_i \in \mathcal{B}_{V_i}$ so that the cartesian product $\prod_{i \in I} A_i$ is not a Borel set in the categorical product $\prod_{i \in I} V_i$. ⊿

◸**2.17.** A measurable space $\mathbb{0}$ is *initial* when there is a unique measurable function $[] : \mathbb{0} \to V$ for every measurable space V. Similarly, a measurable space $\mathbb{1}$ is *terminal* when there is a unique measurable function $\langle\rangle : V \to \mathbb{1}$ for every measurable space V.
(These concepts make sense in every category.)

- Show that **Meas** has exactly one initial space.
- Show that **Meas** has multiple terminal spaces.
- Show that terminal spaces are the product of an empty family of spaces. ⊿

◸**2.18.** Show that each family of subsets generates the σ-algebra of the given space:

- $\{(-\infty, a) | a \in \mathbb{R}\}$, $\{(-\infty, a] | a \in \mathbb{R}\}$, $\{(-\infty, q) | q \in \mathbb{Q}\}$, $\{[a, b) | a, b \in \mathbb{R}\}$ all generate $\mathcal{B}_{\mathbb{R}}$.
- $\{C \cap I_k^n | n \in \mathbb{N}, k \in \mathbf{Fin}\, 2^n\}$ generate the Borel sets of the Cantor space $\mathbb{C}$. What are the corresponding subsets of $\mathbb{T} := 2^{\mathbb{N}}$?
- Show that the set of hemispheres generates the Borel sets of the unit 2-sphere. ⊿

◸**2.19.** Let A be a set. The powerset $\wp A$ is a σ-algebra on A (why?). Define the *discrete* measurable space over A by $\ulcorner A \urcorner := \langle A, \wp A \rangle$. Show:

- For every measurable space V, each function $f : A \to \llcorner V \lrcorner$ is in fact a measurable function $f : \ulcorner A \urcorner \to V$.

The set $\{\emptyset, A\}$ is also a σ-algebra on A (why?). Define the *indiscrete* measurable space over A by $\llcorner \underset{\mathbf{Meas}}{A} \lrcorner := \langle A, \{\emptyset, A\} \rangle$. Show:

- For every measurable space V, each function $f : \llcorner V \lrcorner \to A$ is in fact a measurable function $f : V \to \llcorner \underset{\mathbf{Meas}}{A} \lrcorner$. ⊿

◸**2.20.** Let A be a set and V a measurable space.

- Given a function $f : B \to A$, a subset $X \subseteq B$ is *f-saturated* when $x \in X$ and $fx = fy$ imply $y \in X$. Show that the f-saturated sets form a *topology*:
  - The empty $\varnothing$ set is f-saturated;
  - Arbitrary unions of f-saturated sets are f-saturated; and
  - Finite intersections of f-saturated sets are f-saturated.
  (In fact, arbitrary intersections of f-saturated sets are f-saturated.)
- Show that $f : V \to \ulcorner A \urcorner$ is measurable iff every f-saturated set is measurable.
- Let B be a set and $X_0 \subseteq B$ a subset. We say that a subset $X \subseteq B$ is X_0-*atomic* when $X_0 \subseteq X$ or $X_0 \cap X = \varnothing$. Show that the X_0-atomic subsets are a topology: the empty set is atomic, and finite intersections and arbitrary unions of atomic subsets are atomic.
- Show that $f : \llcorner A_{\mathbf{Meas}} \lrcorner \to V$ is measurable iff all the measurable subsets in V are $\mathrm{Im}\,(f)$-atomic. ◿

◸**2.21.** Let $\ulcorner\mathbb{R}\urcorner := \langle \mathbb{R}, \wp\mathbb{R}\rangle$ be the discrete measurable space over $\mathbb{R}$, and $\tilde{\mathbb{R}}$ be the countable-cocountable measurable space over $\mathbb{R}$. We'll show that the diagonal $\{\langle r, r\rangle | r \in \mathbb{R}\} \subseteq \wp(\mathbb{R} \times \mathbb{R})$ is not a measurable subset of $\ulcorner\mathbb{R}\urcorner \times \tilde{\mathbb{R}}$.
Define the following predicate $\Phi \subseteq \wp(\mathbb{R} \times \mathbb{R})$. Given $K \subseteq \mathbb{R} \times \mathbb{R}$, then $\Phi(K)$ holds when there is a countable sequence of real numbers $\vec{b} \in \mathbb{R}^{\mathbb{N}}$ such that for every $x \in \mathbb{R}$, if there is *some* $y_0 \notin \{b_n | n \in \mathbb{N}\}$ with $\langle x, y_0\rangle \in K$, then *for all* $y \notin \{b_n | n \in \mathbb{N}\}$ we have $\langle x, y\rangle \in K$.
The intuition behind Φ: there is a countable collection of equality constraints on the second component we need to check in order to decide whether a pair is in K. Prove the following.

- $\Phi(K)$ iff there is some $\vec{b} \in \mathbb{R}^{\mathbb{N}}$ and a function $\varphi : \mathbb{R} \to 2^{\mathbb{N}+1}$ such that the indicator function of K is given by:
$$[(x, y) \in K] = \begin{cases} \exists n. y = b_n, \varphi(x, \iota_1 n) = \mathbf{true} : & \mathbf{true} \\ \text{otherwise:} & \varphi(x, \iota_2 \star) \end{cases}$$
- The diagonal $\{\langle r, r\rangle \in \mathbb{R} \times \mathbb{R} | r \in \mathbb{R}\}$ is not in Φ.
- $\Phi(A \times B)$ for every countable and cocountable subset $B \subseteq \mathbb{R}$.
- Φ is closed under countable unions and countable intersections.
- For every measurable subset $K \in \mathcal{B}_{\ulcorner\mathbb{R}\urcorner \times \tilde{\mathbb{R}}}$, both ΦK and $\Phi K^{\complement}$.
- Deduce that the diagonal is not a measurable set in $\ulcorner\mathbb{R}\urcorner \times \tilde{\mathbb{R}}$.

(If you find a shorter proof that the diagonal is not measurable, please let me know!) ◿

3 Basic category theory

We now have enough examples to introduce three important organising concepts from category theory: natural transformations, universal arrows, and adjunctions. This section is aimed at readers who want to take this opportunity to make first steps in category theory, but categorically-savvy readers might also learn some facts about the category of measurable spaces. There's too much material in this section for one sitting, so I recommend reading the first part of each subsection, and referring back to the more advanced parts if you need them later.

3.1 Natural transformations

Ex.2.15 constructs the product of two measurable spaces in the category of measurable spaces. We can record the fact that we can construct this product generally by organising products into a functor. The codomain of this functor is **Meas**, and its domain is the following.

◸**3.1.** Let $\mathbf{Meas}^2$ be the following category:

- Objects are pairs $\vec{X} = \langle X_1, X_2 \rangle$ of measurable spaces.
- Morphisms $\vec{f} : \vec{X} \to \vec{Y}$ are pairs of measurable maps between the corresponding spacse $\vec{f} = \langle f_1 : X_1 \to Y_1, f_2 : X_1 \to Y_1 \rangle$.

There's nothing specific about **Meas** here — we may as well replace it with two generic categories $\mathcal{C}_1, \mathcal{C}_2$ to construct the product category $\mathcal{C}_1 \times \mathcal{C}_2$.

- Spell out the objects and morphisms of $\mathcal{C}_1 \times \mathcal{C}_2$, define identities and composition, and show the resulting structure is a category.
- Define and prove functorial the two projection functors $\boldsymbol{\pi}_i : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_i$.
- Let $\mathcal{C}$ be a category. Define and prove functorial the *diagonal* functor $\boldsymbol{\Delta} : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$. ◿

Binary products organise into a functor $(\times) : \mathbf{Meas}^2 \to \mathbf{Meas}$:

- The action on objects maps each $\vec{X}$ to the binary product $X_1 \times X_2$.
- The action on morphisms maps each $\vec{f} : \vec{X} \to \vec{Y}$ to:

$$f_1 \times f_2 \coloneqq \left\langle X_1 \times X_2 \xrightarrow{\pi_1} X_1 \xrightarrow{f_1} Y_1, \qquad X_1 \times X_2 \xrightarrow{\pi_2} X_2 \xrightarrow{f_2} Y_2 \right\rangle : X_1 \times X_2 \to Y_1 \times Y_2$$

(We apply this functor to pairs of objects and morphisms in infix notation.)

◸**3.2.** Show that $f_1 \times f_2$ is the unique measurable map satisfying for both $i = 1, 2$:

$$\begin{tikzcd}
X_1 \times X_2 \arrow[r, "\pi_i"] \arrow[d, "f_1 \times f_2"'] \arrow[dr, phantom, "="] & X_i \arrow[d, "f_i"] \\
Y_1 \times Y_2 \arrow[r, "\pi_i"'] & Y_i
\end{tikzcd}$$

◿

The equations in the previous exercise characterise the functorial action of the product, and the concept that organises them is that the projections $\pi_i^{\vec{X}} : X_1 \times X_2 \to X_i$ collect into a *natural transformation* $\pi_i : (\times) \to \boldsymbol{\pi}_i$.

In general, let $F, G : \mathcal{B} \to \mathcal{C}$ be functors. The structure of a natural transformation $\alpha : F \to G$, called a *transformation* from F to G is an assignment:

- for each object $X \in \mathcal{B}$, a morphism $\alpha_X : FX \to GX$

The *naturality* property that makes a transformation a natural transformation is:

- for every morphism $f : X \to Y$ in $\mathcal{B}$, we have:

$$\begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] \arrow[dr, phantom, "="] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\alpha_Y"'] & GY \end{tikzcd}$$

◸**3.3.** Let $F, G : \mathbf{Meas}^2 \to \mathbf{Meas}$ are functors whose action on objects maps each $\vec{X}$ to the product $X_1 \times X_2$. Show that if both projections are natural, i.e., for each $i = 1, 2$:

$$\pi_i : F \to \boldsymbol{\pi}_i \qquad \pi_i : G \to \boldsymbol{\pi}_i$$

then F and G have the same action on morphisms. ◿

◸**3.4.** Define the structure and prove the required properties of the following:

- The *identity* functor $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ for every category $\mathcal{C}$.
- The *diagonal* natural transformation $\Delta : \mathrm{Id}_{\mathbf{Meas}} \to (\times)$. ◿

◸**3.5.** Let $\mathbf{Pred}\,\mathbf{Meas} \hookrightarrow \mathbf{Meas}^2$ be the *subcategory* of $\mathbf{Set}^2$:

- Objects are those pairs $\vec{X}$ in which:
  - the points of X_1 are points in X_2: $\llcorner X_1 \lrcorner \subseteq \llcorner X_2 \lrcorner$
  - the σ-algebra on X_1 is the subspace σ-algebra we defined in Ex.2.8.

  So we have a measurable inclusion morphisms we write as $i : X_1 \hookrightarrow X_2$.
- Morphisms are those pairs $\vec{f} : \vec{X} \to \vec{Y}$ for which:

$$\begin{tikzcd} X_1 \arrow[r, hook, "i"] \arrow[d, "f_1"'] \arrow[dr, phantom, "="] & X_2 \arrow[d, "f_2"] \\ Y_1 \arrow[r, hook, "i"'] & Y_2 \end{tikzcd} \tag{1}$$

By stating it is a subcategory, we implicitly define the identities and composition in $\mathbf{Pred}\,\mathbf{Meas}$ by the identities and composition in $\mathbf{Meas}^2$.

- Show that identities and composition are well-defined: identities satisfy the compatibility equation (1).
- Spell out the action of an inclusion functor $\mathbf{Pred}\,\mathbf{Meas} \hookrightarrow \mathbf{Meas}^2$, and show it is indeed functorial, and moreover faithful.
  Since faithful functors reflect categories (Ex.2.6), $\mathbf{Pred}\,\mathbf{Meas}$ is a category.
- Find functors $\mathrm{dom}, \mathrm{cod} : \mathbf{Pred}\,\mathbf{Meas} \to \mathbf{Meas}$ that make the subspace inclusions into a natural transformation $i : \mathrm{dom} \to \mathrm{cod}$. ◿

Let $\mathcal{B}, \mathcal{C}$ be categories. The category $\mathcal{C}^{\mathcal{B}}$ as functors as objects and natural transformations $\alpha : F \to G$ between them as morphisms.

◸**3.6.** Define identities and composition in $\mathcal{C}^{\mathcal{B}}$, faithful *evaluation* functors $\mathrm{eval}(-, X) : \mathcal{C}^{\mathcal{B}} \to \mathcal{C}$ for each $X \in \mathcal{B}$, and a faithful diagonal functor $\boldsymbol{\Delta} : \mathcal{C} \to \mathcal{C}^{\mathcal{B}}$. ◿

◸**3.7.** Let $F, G : \mathcal{B} \to \mathcal{C}$ be functors. Show that a natural transformation $\alpha : F \to G$ is an isomorphism in $\mathcal{C}^{\mathcal{B}}$ iff each $\mathrm{eval}(\alpha, X) \coloneqq \alpha_X : FX \to GX$ is an isomorphism in $\mathcal{C}$. ◿

Every category structure $\mathcal{C}$ has an *opposite* category structure $\mathcal{C}^{\mathrm{op}}$ whose objects are the same, but a morphism from X to Y in $\mathcal{C}^{\mathrm{op}}$ is a morphism from Y to X in $\mathcal{C}$. We will never write morphisms $f : X \to_{\mathcal{C}^{\mathrm{op}}} Y$ in $\mathcal{C}^{\mathrm{op}}$ directly, but instead write them as $f : X \leftarrow Y$.

◸**3.8.** Show that a category structure $\mathcal{C}$ satisfies the defining properties of a category iff its opposite $\mathcal{C}^{\mathrm{op}}$ satisfies them. ◿

◸**3.9.** Let $\mathcal{C}$ be a category.

- Show that $f : X \to Y$ is an isomorphism in $\mathcal{C}$ iff $f : Y \leftarrow X$ is an isomorphism in $\mathcal{C}^{\mathrm{op}}$.
- Show that $\mathbb{1}$ is a terminal object of $\mathcal{C}$ iff $\mathbb{1}$ is an initial object of $\mathcal{C}^{\mathrm{op}}$. ◿

Category theorists use the adverb ‘just’ for this kind of process of unfolding all the structure and comparing the required properties of two concepts. So:

- an isomorphism in $\mathcal{C}^{\mathrm{op}}$ is just an isomorphism in $\mathcal{C}$;
- an initial object in $\mathcal{C}^{\mathrm{op}}$ is just a terminal object in $\mathcal{C}$;
- a natural isomorphism is just a natural transformation consisting of isomorphisms;
- $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ is just $\mathcal{C}$;

and so on. Unlike its colloquial usage, the technical meaning of ‘just’ doesn’t imply this process is simple, obvious, or straightforward. Category theorists tend to forget this difference, which casual listeners sometimes find patronising. If you talk to someone who might not know the technical meaning of ‘just’, try using the more neutral ‘amounts to’.
We define a *contravariant* functor F from $\mathcal{B}$ to $\mathcal{C}$ to be a functor $F : \mathcal{B}^{\mathrm{op}} \to \mathcal{C}$.

◸**3.10.** Show that contravariant functors:

- Reflect categories when faithful.
- Preserve isomorphisms.
- Reflect isomorphism pairs when faithful. ◿

◸**3.11.** A functor $H : \mathcal{B} \to \mathcal{C}$ is *fully-faithful* when its action on morphisms is bijective: for every morphism $g : HX \to HY$ there is a unique morphism $f : X \to Y$ such that $Hf = g$. Show that fully-faithful functors *lift* isomorphic objects: if $H : \mathcal{B} \to \mathcal{A}$ is fully-faithful and $g : HA \xrightarrow{\cong} HB$ is an isomorphism, then there is an isomorphism $f : A \xrightarrow{\cong} B$ and H maps it to g. ◿

We’ll now define the most important functor in category theory. Let $\mathcal{C}$ be a *locally small* category: each collection of morphisms from X to Y is a set $\mathcal{C}(X, Y)$ in our universe of sets. We then have the following functor $\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$:

- Its action on objects sends a pair of objects to the set of morphisms between them: $\mathrm{Hom}_{\mathcal{C}} \langle X, Y \rangle \coloneqq \mathcal{C}(X, Y)$.
- Its action on morphisms precomposes the contravariant argument and postcomposes the covariant argument:

$$\mathrm{Hom}_{\mathcal{C}} \langle f : X_1 \leftarrow X_2, g : Y_1 \to Y_2 \rangle : \left(X_1 \xrightarrow{u} Y_1\right) \mapsto \left(X_2 \xrightarrow{f} X_1 \xrightarrow{u} Y_1 \xrightarrow{g} Y_2\right)$$

We'll write $\mathcal{C}(x, y)$ for $\mathrm{Hom}_{\mathcal{C}}\langle x, y\rangle$ for morphisms as well as objects. This notation matches previous conventions, like the product functor, where we used the same notation for morphisms and objects.

◸**3.12.** Show that $\mathrm{Hom}_{\mathcal{C}}$ is a functor. Show that its curried version $\mathbf{y}_{\mathcal{C}} : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ is also a functor. It is called the *Yoneda embedding*. Show that the alternative currying $\mathbf{y}' : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}^{\mathcal{C}}$ is just $\mathbf{y}_{\mathcal{C}^{\mathrm{op}}} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}^{(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}}$ for the opposite category. ◿

Because the iterated superscripts are hard to read, you'll see the notation $\hat{\mathcal{C}} := \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$.

◸**3.13.** Let $F : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$ be a functor from a *small* category $\mathbb{C}$: a category with a set of objects and a set of morphisms.

- Type-check that $\lambda x.\mathrm{Hom}_{\mathbf{Set}^{\mathbb{C}^{\mathrm{op}}}}\langle \mathbf{y}x, F\rangle : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$, which we may write as $\lambda x.\hat{\mathcal{C}}(\mathbf{y}x, F)$.
- Prove the *Yoneda lemma*: the operation 'evaluate each natural transformation at the identity morphism' is a natural isomorphism $\Upsilon : \left(\lambda x.\hat{\mathbb{C}}(\mathbf{y}x, F)\right) \xrightarrow{\cong} F$.
- Show that $\mathbf{y} : \mathbb{C} \to \hat{\mathbb{C}}$ is fully-faithful. ◿

3.2 Universality and representability

Universality lets us pin-point what makes a construction special. Let $H : \mathcal{B} \to \mathcal{C}$ be a functor, and $A \in \mathcal{C}$ an object. An *arrow from A to H* is a pair $\langle X, f\rangle$ consisting of:

- an object X in $\mathcal{B}$; and
- a morphism $f : A \to HX$ in $\mathcal{C}$.

An *arrow morphism* $h : \langle X, f\rangle \to \langle Y, g\rangle$ is a morphism $h : X \to Y$ satisfying:

$$\begin{tikzcd} & HX \arrow[dd, "Hh"] \\ A \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[r, phantom, "="] & \\ & HY \end{tikzcd}$$

Arrows from A to H and their morphisms form a category. A *universal arrow from A to H* is an initial object in this category.

◸**3.14.** Define the remaining structure of the category of arrows from A to H. Define a faithful functor from this category structure to $\mathcal{B}$. ◿

◸**3.15.** Let A be a set. Find a universal arrow from A to the functor $\llcorner - \lrcorner : \mathbf{Meas} \to \mathbf{Set}$. ◿

◸**3.16.** Let V be a measurable space. Find a universal arrow from V to the functor $\mathrm{cod} : \mathbf{Pred\,Meas} \to \mathbf{Meas}$ you defined in Ex.3.5. ◿

We define arrows from H to A similarly, as pairs $\langle X, f\rangle$ where $f : HX \to A$, and morphisms:

$$\begin{tikzcd} HX \arrow[dd, "Hh"'] \arrow[dr, "f"] & \\ \arrow[r, phantom, "="] & A \\ HY \arrow[ur, "g"'] & \end{tikzcd}$$

A universal arrow from H to A is then a terminal arrow in this category.

◸**3.17.** Find a universal arrow from the functor $\llcorner - \lrcorner : \mathbf{Meas} \to \mathbf{Set}$ to a set A. ◿

◸**3.18.** Find a universal arrow from the diagonal functor $\mathbf{\Delta} : \mathbf{Meas} \to \mathbf{Meas}^2$ to a pair of measurable spaces $\vec{X}$. ◿

◸**3.19.** Let A be a set. A *global geometry* $\mathcal{G}$ on A is a family of sets $\mathcal{G} \subseteq \wp A$. A *globally geometric space* X is then a pair $\langle \llcorner X \lrcorner, \mathcal{G}_X \rangle$ consisting of a set $\llcorner X \lrcorner$ of points and a global geometry $\mathcal{G}_X \subseteq \wp \llcorner X \lrcorner$. Given two globally geometric spaces X, Y, a *globally geometric morphism* $f : X \to Y$ is a function $f : \llcorner X \lrcorner \to \llcorner Y \lrcorner$ such that, for every subset in the codomain geometry $U \in \mathcal{G}_Y$, its inverse image is in the source geometry $f^{-1}[U] \in \mathcal{G}_X$.

- Define the structure of a category **Geom** whose objects are globally geometric spaces and their morphisms, and a faithful functor $\llcorner - \lrcorner : \mathbf{Geom} \to \mathbf{Set}$.
- Let A be a set. Find universal arrows from A to $\llcorner - \lrcorner$ and from $\llcorner - \lrcorner$ to A.
- Each σ-algebra is a global geometry, yielding a faithful functor $\llcorner - \lrcorner_{\mathbf{Geom}} : \mathbf{Meas} \hookrightarrow \mathbf{Geom}$. Let X be a globally geometric space. Find a universal arrow from $\llcorner - \lrcorner_{\mathbf{Geom}}$ to X. ⊿

◸**3.20.** Let A be a set. Let $\mathbf{Rel}_A$ be the following category:

- objects are binary relations R over A, i.e.: $R \subseteq A \times A$; and
- there is a unique morphisms $f : R \to S$ when $R \subseteq S$.

Let $\llcorner - \lrcorner : \mathbf{Equiv}_A \hookrightarrow \mathbf{Rel}_A$ be the subcategory consisting of the equivalence relations and its associated faithful functor.
For every relation R, find a universal arrow from R to $\llcorner - \lrcorner$. ⊿

Let I, $\mathcal{C}$ be categories. A *diagram of shape I in $\mathcal{C}$* is a functor $D : I \to \mathcal{C}$. A morphism $\alpha : D \to E$ between diagrams is a natural transformation. The functor category $\mathcal{C}^I$ then serves as the category of diagrams and their morphisms.
A *cone* for a diagram $D : I \to \mathcal{C}$ is a pair $\langle C, c \rangle$ consisting of:

- an object $C \in \mathcal{C}$, called the vertex of the cone; and
- a natural transformation $c : \mathbf{\Delta} C \to D$, i.e., an assignment for each $i \in I$ of a morphism $C \to Di$ in $\mathcal{C}$ such that for every $u : i \to j$ in I, we have:

$$\begin{tikzcd}
& Di \arrow[dd, "Du"] \\
C \arrow[ur, "c_i"] \arrow[dr, "c_j"'] \arrow[r, phantom, "="] & \\
& Dj
\end{tikzcd}$$

A *cone morphism* $h : \langle B, b \rangle \to \langle C, c \rangle$ is a morhpism $h : B \to C$ satisfying, for all $i \in I$:

$$\begin{tikzcd}
& & Di \\
B \arrow[r, "h"'] \arrow[urr, bend left, "b_i"] & C \arrow[ur, "c_i"'] \arrow[u, phantom, "="] &
\end{tikzcd}$$

◸**3.21.** Show that a D-cone is just an arrow from the diagonal functor $\mathbf{\Delta} : \mathcal{C} \to \mathcal{C}^I$ to the diagram $D \in \mathcal{C}^I$. ⊿

◸**3.22.** Find a category $\mathbf{2}$ so that the diagram category $\mathbf{Meas}^{\mathbf{2}}$ is just $\mathbf{Meas}^2$. ⊿

A *limiting cone* is a universal cone, that is, a universal arrow from $\mathbf{\Delta}$ to D. Its vertex is called a *limit* of the diagram. Similarly, a *colimiting cocone* is a universal arrow from D to $\mathbf{\Delta}$, and its vertex is called the *colimit* of the diagram.

◸**3.23.** Show that a terminal object is just a limiting cone for the diagram from the category with no objects and no morphisms. ⊿

◸**3.24.** Show that a limit in $\mathbf{Rel}A$ is just the intersection of the relations in the diagram, and a colimit is just the union. ◿

◸**3.25.** Let I be the category with two objects 0, 1 and four morphisms:

- The two identities: id_0, id_1; and
- $f : 0 \to 1$ and $g : 1 \to 0$.

Define composition to satisfy the neutrality axioms whenever an identity is involved, and in the remaining cases define:

$$f \circ g \coloneqq \mathrm{id}_1 \qquad g \circ f \coloneqq \mathrm{id}_0$$

- Define a faithful functor $U : I \to \mathbf{Set}$ sending 0 to $\{0\}$ and 1 to $\{1\}$ and deduce I is indeed a category.
- Show that a diagram $D : I \to \mathcal{C}$ is just an isomorphism pair. ◿

◸**3.26.** Let $D : I \to \mathbf{Set}$ be a *small* diagram — a diagram whose domain I is a small category.

- Define $L \coloneqq \{\vec{x} \in \prod_{i \in I} Di | \forall u : i \to j \in I . x_j = Dux_i\}$ and $\ell_i : L \to Di$ to be the restriction of the i-th component projection. Show that $\langle L, \ell\rangle$ is a limiting cone for D.
- Let R to be the relation on the disjoint union $\coprod_{i \in I} Di$ given by $\langle i, x\rangle \, R \, \langle j, y\rangle$ when there is some $u : i \to j$ with $Dux = y$. Let $\equiv_R$ be the reflexive-transitive-symmetric closure of $\equiv_R$. Define a cocone by setting $C \coloneqq \coprod_{i \in I} Di / \equiv_R$ and c_i mapping each $x \in Di$ to $[\langle i, x\rangle]$, the $\equiv_R$-equivalence class of $\langle i, x\rangle$. Show that $\langle C, c\rangle$ is a colimiting cocone for D. ◿

Let $\mathcal{D}$ be a class of diagrams in a category $\mathcal{C}$. We say that $\mathcal{C}$ is $\mathcal{D}$-*complete* when it has limit cones for all diagrams in $\mathcal{D}$, and $\mathcal{D}$-*cocomplete* when it has colimiting cocones for all diagrams in $\mathcal{D}$. By default, $\mathcal{D}$ is the class of all small diagrams. The category **Set** is therefore complete and cocomplete. We can often use this fact to transfer limits and colimits along functors into other categories.
Let $H : \mathcal{B} \to \mathcal{C}$ be a functor, and $D : I \to \mathcal{B}$ a diagram. If $c : \mathbf{\Delta} C \to D$ is a D-cone, then $Hc : \mathbf{\Delta} HC \to H \circ D$ is an $H \circ D$-cone. We say that H:

- preserves D-limits when, for every limiting cone $\langle L, \ell\rangle$, the cone $\langle HL, H\ell\rangle$ is limiting for $H \circ D$;
- reflects D-limits when, for every D-cone $\langle L, \ell\rangle$, if the cone $\langle HL, H\ell\rangle$ is $H \circ D$-limiting, then $\langle L, \ell\rangle$ is limiting (and then H preserves this limit); and
- lifts D-limits when, for every $H \circ D$-cone $\langle L', \ell'\rangle$ there is a D-limiting cone $\langle L, \ell\rangle$ and a cone isomorphism $\langle HL, H\ell\rangle \cong \langle L', \ell'\rangle$.

We extend these to a class of diagrams $\mathcal{D}$ by saying that H preserves/reflects/lifts $\mathcal{D}$-limits of the class if it does so for each diagram in $\mathcal{D}$. Finally, we say that:

- H is $\mathcal{D}$-continuous when it preserves $\mathcal{D}$-limits;
- H *generates* $\mathcal{D}$-limits when it preserves and lifts $\mathcal{D}$-limits; and
- H *creates* $\mathcal{D}$-limits when it preserves, reflects, and lifts $\mathcal{D}$-limits.

We define analogous concepts for colimits.

◸**3.27.** Show that $\llcorner - \lrcorner : \mathbf{Meas} \to \mathbf{Set}$ lifts limits, but does not reflect limits. ◿

◸**3.28.** Show that if $H : \mathcal{B} \to \mathcal{C}$ lifts $\mathcal{D}$-limits and $\mathcal{C}$ is $\mathcal{D}$-complete, then $\mathcal{B}$ is also $\mathcal{D}$-complete and H generates $\mathcal{D}$-limits. Deduce that **Meas** is complete. ⊿

◸**3.29.** Show that the Yoneda embedding preserves limits. ⊿

Let $\mathcal{C}$ be a locally small category. A functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is *representable* when there is some object X and a natural isomorphism $\rho : \mathbf{y}X \xrightarrow{\cong} F$. We call the object X the *representing object* and the isomorphism ρ the *representation.*

◸**3.30.** Let $H : \mathcal{B} \to \mathcal{C}$ be a functor between locally small categories. Show that a universal arrow $\langle X, f\rangle$ from A to H is just a representation $\rho : \mathbf{y}_{\mathcal{B}^{\mathrm{op}}} X \xrightarrow{\cong} \lambda x.\mathcal{C}(A, Hx)$, and the translation between f and ρ is given by the Yoneda lemma:

$$f \in \mathcal{C}(A, HX) \qquad \xleftrightarrow{\Upsilon_X} \qquad \rho \in \widehat{\mathcal{B}^{\mathrm{op}}}(\mathbf{y}X, \lambda x.\mathcal{C}(A, Hx))$$ ⊿

Solution. A representation ρ is a family of bijections, natural in Y, between two hom-sets:

$$\frac{A \longrightarrow HY}{X \longrightarrow Y} \tag{2}$$

The Yoneda lemma gives an arrow $f \coloneqq \Upsilon\rho$. The naturality of ρ implies that for all $h : X \to Y$:

$$\begin{tikzcd}
\mathcal{B}(X,X) \arrow[r, "\rho_X"] \arrow[d, bend right, "\mathcal{B}(X,h)"'] \arrow[d, bend left, "\mathbf{y}_{\mathcal{B}^{\mathrm{op}}} h"] \arrow[d, phantom, "="] \arrow[dr, phantom, "="] & \mathcal{C}(A,HX) \arrow[d, "\mathcal{C}(A,Hh)"] \\
\mathcal{B}(X,Y) \arrow[r, "\rho_Y"'] & \mathcal{C}(A,HY)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{id}_X \arrow[r, mapsto, "\rho_X"] \arrow[d, mapsto, "\mathcal{B}(X,h)"'] \arrow[dr, phantom, "="] & \rho_X(\mathrm{id}_X) = \Upsilon_X \rho =: f \arrow[d, mapsto, "\mathcal{C}(A,Hh)"] \\
h \circ \mathrm{id}_X = h \arrow[r, mapsto, "\rho_Y"'] & \rho_Y h = Hh \circ f
\end{tikzcd}$$

So the bijection ρ_Y acts by $\lambda h.Hh \circ f$, and that's the universality of the arrow $\langle X, f\rangle$.
Conversely, a universal arrow $\langle X, f\rangle$ induces a bijective correspondences as in (2) given by $\rho_Y(h : X \to Y) \coloneqq Hh \circ f = \mathcal{C}(A, Hh)f = (\Upsilon_X^{-1} f)_Y h$, and so $\rho = \Upsilon_X^{-1} f$, and also Y-natural. ◢

3.3 Adjunctions

The input to the universal arrow concept in the previous section is a functor and an object A of $\mathcal{C}$. By currying the object, we arrive at the concept of an adjoint:

- Let $U : \mathcal{B} \to \mathcal{C}$ be a functor. A *left adjoint to* U, $\langle FA \in \mathcal{B}, \eta_A : A \to U(FA)\rangle_{A \in \mathcal{C}}$, is an assignment, for each object $A \in \mathcal{C}$, of a universal arrow $\langle FA, \eta_A\rangle$ from A to U.
- Similarly, let $F : \mathcal{C} \to \mathcal{B}$ be a functor. A *right adjoint to* F, $\langle UX, \varepsilon_X : F(UX) \to X\rangle_{X \in \mathcal{B}}$, is an assignnment, for each object $X \in \mathcal{B}$, of a universal arrow $\langle UX, \varepsilon_X\rangle$.

So an adjoint is a simultaneous assignment of universal arrows.
So far we've seen plenty of examples of adjoints:

◸**3.31.** Show that the functors $\llcorner - \lrcorner : \mathbf{Meas} \to \mathbf{Set}$ and $\llcorner - \lrcorner : \mathbf{Geom} \to \mathbf{Set}$ have both a left and right adjoints. ⊿

◸**3.32.** Show that the functor $\llcorner - \lrcorner_{\mathbf{Geom}} : \mathbf{Meas} \hookrightarrow \mathbf{Geom}$ has a right adjoint. ⊿

◸**3.33.** Show that the functor $\llcorner - \lrcorner : \mathbf{Equiv}_A \hookrightarrow \mathbf{Rel}_A$ has a left adjoint. ◿

◸**3.34.** Show that the diagonal functor $\boldsymbol{\Delta} : \mathbf{Meas} \to \mathbf{Meas}^2$ has a right adjoint. ◿

◸**3.35.** Show that every diagonal functor $\boldsymbol{\Delta} : \mathbf{Set} \to \mathbf{Set}^I$ for a small category I, has both a left and a right adjoint. Every diagonal functor $\boldsymbol{\Delta} : \mathbf{Meas} \to \mathbf{Meas}^I$ for a small category I has a right adjoint. ◿

Let $U : \mathcal{B} \to \mathcal{C}$ be a functor with a right adjoint $\langle F, \eta\rangle$. By Ex.3.30, each universal arrow $\langle FA, \eta_A\rangle$ comes from a representation $\rho_A : \mathbf{y}(FA) \xrightarrow{\cong} \lambda x.\mathcal{C}(A, Ux)$, so we have a collection of bijections, indexed by both $A \in \mathcal{C}$ and $X \in Y$:

$$\rho_{A,X} : \mathcal{B}(FA, X) \xrightarrow{\cong} \mathcal{C}(A, UX) \qquad \rho_{A,X} : \left(FA \xrightarrow{h} X\right) \mapsto \left(A \xrightarrow{\eta_A} U(FA) \xrightarrow{Uh} UX\right)$$

It is natural in X, but if we want it to be natural in A, we need to equip F with a functorial action on morphisms.

◸**3.36.** Show that there is exactly one action on morphisms such that:

- $F : \mathcal{C} \to \mathcal{B}$ is a functor; and
- $\rho : (\lambda x, y.\mathcal{B}(Fx, y)) \xrightarrow{\cong} (\lambda x, y.\mathcal{C}(x, Uy))$ is a natural transformation (and so forms a natural isomorphism). ◿

An *adjunction* from $\mathcal{C}$ to $\mathcal{B}$ is a tuple $\langle F, G, \rho\rangle$ consisting of:

- Two functors $F : \mathcal{C} \to \mathcal{B}$, the *left adjoint* and $G : \mathcal{B} \to \mathcal{C}$, the *right adjoint*; and
- A natural isomorphism $\rho : (\lambda x, y.\mathcal{B}(Fx, y)) \xrightarrow{\cong} (\lambda x, y.\mathcal{C}(x, Uy))$ called the *mate bijection*.

By Ex.3.36, each adjoint extends to a unique adjunction. This process exhausts all adjunctions. Indeed, in an adjunction, each bijection $\rho_A : \mathbf{y}FA \xrightarrow{\cong} \lambda y.\mathcal{C}(A, y)$ is a representation. By Ex.3.30, setting $\eta_A \coloneqq \Upsilon_{FA}\rho_A : A \to UFA$ gives a simultaneous assignment $\langle FA, \eta_A\rangle_{A \in \mathcal{C}}$ of universal arrows from A to U, i.e., a left adjoint to U. Using a similar argument for right adjoints, we have that an adjunction is just an adjoint (left or right) to the appropriate functor in the adjunction.
Given an adjunction, as we've seen, the universal arrows are given by:

$$\eta_A \coloneqq \rho_{A,FA}\mathrm{id}_{FA} = \Upsilon_{FA}(\rho_A) \qquad \varepsilon_X \coloneqq \rho^{-1}_{UX,X}\mathrm{id}_{UX} = \Upsilon_{UX}(\rho^{-1}_X)$$

◸**3.37.** Show that $\eta : \mathrm{Id}_{\mathcal{C}} \to U \circ F$ and $\varepsilon : F \circ U \to \mathrm{Id}_{\mathcal{B}}$ are natural. Can you do it by appealing to the naturality of the Yoneda lemma? ◿

Solution. We'll do so for η, the proof for ε is similar. We need to show, for $f : A \to B$ in $\mathcal{C}$:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & UFA \arrow[d, "UFf"] \\ B \arrow[r, "\eta_B"'] & UFB \end{tikzcd}$$

The Yoneda lemma in question is the natural isomorphism:

$$\Upsilon : \left(\lambda x.\hat{\mathcal{B}}(\mathbf{y}x, \lambda y.\mathcal{C}(A, Uy))\right) \xrightarrow{\cong} (\lambda x.\mathcal{C}(A, Ux)$$

$$
\begin{tikzcd}
\rho_A \arrow[rrr, mapsto] \arrow[ddd, mapsto, bend right] & & & \eta_A \arrow[ddd, mapsto, bend left] \\
& \hat{\mathcal{B}}(\mathbf{y}FA, \lambda y.\mathcal{C}(A,Uy)) \arrow[r, "\Upsilon_{FA}"] \arrow[d, "\hat{\mathcal{B}}(\mathbf{y}(Ff), \lambda y.\mathcal{C}(A,Uy))"'] \arrow[dr, phantom, "="] & C(A,UFA) \arrow[d, "\mathcal{C}(A,UFf)"] & \\
& \hat{\mathcal{B}}(\mathbf{y}FB, \lambda y.\mathcal{C}(A,Uy)) \arrow[r, "\Upsilon_{FB}"'] & C(A,UFB) & \\
\underset{\substack{\uparrow \\ \rho\text{-naturality}}}{(\lambda h.\rho_A(h\circ Ff)) = (\lambda h.\rho_B h\circ f)} \arrow[rrr, mapsto] & & & \eta_B\circ f = UFf\circ\eta_A
\end{tikzcd}
$$

as we wanted. ◢

We have $\varepsilon_{FA}\circ F\eta_A = \rho^{-1}_{FA}(\eta_A) = \mathrm{id}_{FA}$ and similarly $U\varepsilon_X\circ\eta_{UX} = \mathrm{id}_{UX}$, and we arrived at the following concept.

A *formal adjunction* $\langle F, G, \eta, \varepsilon\rangle$ consists of:

- Two functors $F : \mathcal{C}\to\mathcal{B}$, the *left adjoint* and $G : \mathcal{B}\to\mathcal{C}$, the *right adjoint*; and
- Two natural transformations $\eta : \mathrm{Id}_{\mathcal{C}}\to U\circ F$, the *unit*, and $\varepsilon : F\circ U\to\mathrm{Id}_{\mathcal{B}}$, the *counit*.

satisfying the following *triangle equalities*, for every $X\in\mathcal{B}$ and $A\in\mathcal{C}$:

$$
\begin{tikzcd}
FA \arrow[rr, equal] \arrow[dr, "F\eta_A"'] & \arrow[d, phantom, "="] & FA \\
& FUFA \arrow[ur, "\varepsilon_{FA}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& UFUX \arrow[dr, "U\varepsilon_X"] \arrow[d, phantom, "="] & \\
UX \arrow[ur, "\eta_{UX}"] \arrow[rr, equal] & \; & UX
\end{tikzcd}
$$

The term 'formal' here is used in the Australian sense: it involves categories, functors (morphisms between categories), and natural transformations (morphisms between functors), and so can be generalised to 'formal' categories that have 0-cells (objects), 1-cells (morphisms between 0-cells), and 2-cells (morphisms between 1-cells).

We've shown that every adjunction gives rise to a formal adjunction. The mate isomorphism ρ is determined by η as:

$$\rho_{A,X}h \coloneqq Hh\circ\eta_A = (\Upsilon^{-1}_{FA}\eta_A)_X h \tag{3}$$

Therefore, this formal adjunction is determined uniquely. To see this process is exhaustive, take any formal adjunction, and set $\rho_A \coloneqq \Upsilon^{-1}_{FA}\eta_A : \mathbf{y}FA\to\lambda y.C(A,Uy)$ using the Yoneda lemma as in (3). Then ρ_A is natural.

◸**3.38.** Show that ρ_A is an isomorphism. ◿

Solution. Define $\rho^{-1}_{A,X}h' \coloneqq \varepsilon_X\circ Fh'$ and show it is inverse to $\rho_{A,X}$ by direct calculation. For example, take any $h' : A\to UX$ in $\mathcal{C}$, and show that $h = \rho_{A,X}(\rho^{-1}_{A,X}h) = U\varepsilon_X\circ UFh'\circ\eta$:

$$
\begin{tikzcd}
A \arrow[rr, "\eta_A"] \arrow[dr, "h'"] \arrow[dd, "h'"'] & \arrow[d, phantom, "\substack{\eta\text{-nat}\\ =}"] & UFA \arrow[dd, "UFh'"] \\
\arrow[r, phantom, "\substack{\mathrm{id}\\ =}"] & UX \arrow[dl, equal] \arrow[dr, "\eta_{UX}"] \arrow[d, phantom, "\substack{\text{triangle}\\ =}"] & \\
UX & \; & UFUX \arrow[ll, "U\varepsilon"]
\end{tikzcd}
$$

as we wanted. ◢

Since ρ_A is a natural isomorphism, by Ex.3.30, we have a universal arrow $\langle FA, \eta_A \rangle$ from A to U, so we have a left adjoint to U, hence an adjunction.

◸**3.39.** Check that the resulting formal adjunction is our given formal adjunction. ⊿

Summarising, a formal adjunction is just an adjunction, which in turn is both just a left adjoint F to a functor $U : \mathcal{B} \to \mathcal{C}$ and just a right adjoint U to a functor $F : \mathcal{C} \to \mathcal{B}$. We write $\rho, \langle \eta, \varepsilon \rangle : F \dashv U : \mathcal{B} \to \mathcal{C}$ where ρ is the mate isomorphism of the adjunction, and η is the unit and ε the counit of the formal adjunction.

◸**3.40.** Let $\langle \eta, \varepsilon \rangle : F \dashv U : \mathcal{B} \to \mathcal{C}$ and let $H : \mathcal{C} \to \mathcal{D}$. Show that if $\langle A, v \rangle$ is a universal arrow from V to H, then $\left\langle FA, V \xrightarrow{v} HX \xrightarrow{H\eta_A} HU(FX) \right\rangle$ is a universal arrow from V to $H \circ U$. Deduce that given two composable adjunctions $F_1 \dashv U_1$ and $F_2 \dashv U_2$:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "F_2"] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, bend left, "U_2"] \arrow[r, bend left, "F_1"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "U_1"]
\end{tikzcd}$$

their composition is also an adjunction $F_1 \circ F_2 \dashv U_2 \circ U_1$. ⊿

◸**3.41.** Show that right adjoints preserve limits and left adjoints preserve colimits. ⊿

4 Aumann's theorem

These exercises explore concepts derived from and around Aumann's theorem. We will not need intimate knowledge of the Borel hierarchy, but if you're curious about it, the exercises in Sec. A explore it in further detail through. This section is also an opportunity to learn and practice some category theory.

Let X, Y be measurable spaces. An *exponential of* Y *by* X is a pair $\left\langle Y^X, \mathrm{eval}\right\rangle$ consisting of a measurable space Y^X and a measurable function $\mathrm{eval} : Y^X \times X \to Y$ such that for every measurable space Γ and measurable function $f : \Gamma \times X \to Y$ there exists a unique measurable function $\lambda f : \Gamma \to Y^X$ satisfying:

$$\begin{tikzcd} \Gamma \arrow[d, dashed, "\lambda f"'] \\ Y^X \end{tikzcd} \qquad \begin{tikzcd} \Gamma \times X \arrow[d, "(\lambda f) \times \mathrm{id}_X"'] \arrow[dr, "f"] \arrow[dr, phantom, "=" near start, shift right=3] & \\ Y^X \times X \arrow[r, "\mathrm{eval}"'] & Y \end{tikzcd}$$

This definition is a standard category-theoretic notion — we could replace 'measurable space' by 'object' and 'measurable function' by 'morphism', as long as the category has products with X.

◸**4.1.** Let I be a countable set and Y a measurable space. Show that we can give an exponential of Y the discrete measurable space over I by the product $Y^{\ulcorner I \urcorner} := \prod_{i \in I} X$. Where in your proof do you use I's countability? ◿

◸**4.2.** Let $\left\langle Y^X, \mathrm{eval}\right\rangle$ be an exponential in **Meas**.

- Find a bijection between the points in Y^X and the measurable functions from X to Y, that is: $\llcorner Y^X \lrcorner \cong \mathbf{Meas}(X, Y)$
- Show that there is a σ-algebra on $\mathbf{Meas}(X, Y)$ such that the set-theoretic evaluation function $\mathrm{eval} : \mathbf{Meas}(X, Y) \times X \xrightarrow{\langle f, x\rangle \mapsto f(x)} Y$ is measurable. ◿

◸**4.3.** Let I be a set.
Let $\langle X_i \rangle_{i \in I}$ be an I-indexed family of measurable spaces. Their *coproduct* $\langle \coprod_{i \in I} X_i, \iota_{-} \rangle$ consists of the measurable space $\coprod_{i \in I} X_i$ whose:

- points are pairs of a tag from I and a point from X_i:

$$\llcorner \coprod_{i \in I} X_i \lrcorner := \coprod_{i \in I} \llcorner X_i \lrcorner := \bigcup_{i \in I} \{i\} \times \llcorner X_i \lrcorner$$

- measurable subsets are unions $\bigcup_{i \in I} \{i\} \times U_i$ of arbitrary I-indexed family of measurable subsets $U_i \in \mathcal{B}_{X_i}$.

and for each $i \in I$, $\iota_i : X_i \xrightarrow{x \mapsto \langle i, x\rangle} \coprod_{i \in I} X_i$.
Prove:

- $U \subseteq \coprod_{i \in I} \llcorner X_i \lrcorner$ is measurable iff $\iota_i^{-1}[U]$ is measurable for all $i \in I$.
- The σ-algebra axioms hold in the coproduct, and every injection is measurable.
- For every I-indexed family of measurable functions $f_i : X_i \to Y$ there is a unique measurable function $[f_i]_{i \in I} : \coprod_{i \in I} X_i \to Y$ such that:

$$\begin{tikzcd} X_i \arrow[r, "\iota_i"] \arrow[dr, "f_i"'] & \coprod_{i\in I} X_i \arrow[d, dashed, "{[f_i]_{i\in I}}"] \arrow[dl, phantom, "="] \\ & Y \end{tikzcd}$$

- Find, and show the uniqueness of, the functorial action that makes the coproduct construction into a functor $\coprod_I : \mathbf{Meas}^I \to \mathbf{Meas}$ and all the coproduct injections natural transformations $\iota_i : \boldsymbol{\pi}_i \to \coprod_I$. ◿

◸**4.4.** We say that a space X is *exponentiable* when there is an exponential Y^X for every measurable space Y.

- Let I be a set. Show that if, for every measurable space Γ, the following canonical map is a measurable isomorphism: $\coprod_{i\in I}\Gamma \xrightarrow{[\langle \mathrm{id}_\Gamma, i\rangle]_{i\in I}} \Gamma \times \ulcorner I \urcorner$, then $\ulcorner I \urcorner$ is exponentiable, and $\langle \prod_{i\in I} Y, \langle \vec{x}, i\rangle \mapsto x_i\rangle$ is the exponential $\left\langle Y^{\ulcorner I \urcorner}, \mathrm{eval}\right\rangle$ of Y by $\ulcorner I \urcorner$.
- Show, for every countable set I, that $\ulcorner I \urcorner$ is exponentiable.
- Show that if X is exponentiable, then for every I-indexed family of spaces, the canonical map $\coprod_{i\in I} X \to X \times \ulcorner I \urcorner$ is a measurable isomorphism. ◿

Aumann's theorem shows that **Meas** cannot have an exponential for $\mathbb{R}$ by $\mathbb{R}$ by inspecting the full Borel hierarchy of the product. The next few exercises explore a more elementary example for two measurable spaces that don't have an exponential. I learned of this example from Christine Tasson and Johannes Hölzl.

◸**4.5.** Consider the following measurable spaces:

- $\ulcorner \mathbb{R} \urcorner := \langle \mathbb{R}, \wp\mathbb{R}\rangle$: the discrete measurable space over the real numbers.
- $\tilde{\mathbb{R}}$: the measurable space over the real numbers with the countable-cocountable σ-algebra.
- $\mathbb{2} := \mathbf{Fin}\,2 := \{\mathbf{true} := 1, \mathbf{false} := 0\}$: the discrete space with two points.

We'll show that the exponential $\mathbb{2}^{\tilde{\mathbb{R}}}$ doesn't exist in **Meas**.

- Show that the diagonal $\{\langle r, r\rangle \in \mathbb{R}\times\mathbb{R} | r \in \mathbb{R}\}$ is a measurable subset of $\coprod_{r\in\mathbb{R}} \tilde{\mathbb{R}}$, and deduce that $\ulcorner \mathbb{R} \urcorner$ is not exponentiable.
  (This fact doesn't tell us which space Y doesn't have the exponential $Y^{\ulcorner \mathbb{R} \urcorner}$.)
- Show that if we have an exponential $\mathbb{2}^{\tilde{\mathbb{R}}}$, then the curried diagonal is a measurable function $\lambda r.\lambda s.[r = s] : \ulcorner \mathbb{R} \urcorner \to \mathbb{2}^{\tilde{\mathbb{R}}}$. ◿

Aumann's theorem is still worth the effort. The spaces in the previous exercise may seem pathological, and we may falsely hope to exclude them by restricting to a subcategory of 'nice' spaces. Aumann's theorem concerns indispensable spaces: $\mathbb{2}$ and $\mathbb{R}$.
A frequent reaction to Aumann's theorem is to hope that we can avoid it by replacing the set of Borel measurable functions with a larger set of functions $f : \mathbb{R} \to \mathbb{R}$, such as the Lebesgue-measurable functions, or the universally measurable functions. This is not the case. Here's an 'easy', but unsatisfying, result:

◸**4.6.** Let E be a measurable space consisting of a σ-algebra over a set of functions that contains all the Borel measurable functions: $\mathbf{Meas}(\mathbb{R},\mathbb{R}) \subseteq \llcorner E \lrcorner \subseteq \mathbf{Set}(\mathbb{R},\mathbb{R})$. Show that the evaluation function $\mathrm{eval} : E \times \mathbb{R} \to \mathbb{R}$ is not measurable. ◿

This result is unsatisfying because the σ-algebra on $\mathbb{R}$ in the domain of eval is the Borel one, so if we dare to include even one non-Borel-measurable function $f : \mathbb{R} \to \mathbb{R}$, then eval $\langle f, -\rangle : \mathbb{R} \to \mathbb{R}$ won't be measurable. The convincing result is that even if we take S to be the real numbers together with the much bigger σ-algebra of *Lebesgue*-measurable sets, then we still don't have any σ-algebra on the *Borel*-measurable functions that makes the evaluation function $\mathrm{eval} : \mathbf{Meas}(\mathbb{R}, \mathbb{R}) \times S \to \mathbb{R}$ measurable. Doing so will require us to define the Lebesgue measurable sets, which will take us deeper into classical measure theory. This price is a hefty one to pay for just a dead-end, so I moved this material to Sec. B .If you're curious, jump right ahead.

5 Sequences

◸**5.1.** Show that the following sets are Borel in the extended real numbers $[-\infty, \infty]$:

- The set of converging sequences (including sequences whose limit $\pm\infty$):

$$\mathrm{Converge}[-\infty,\infty] \coloneqq \left\{\vec{r} \in [-\infty,\infty]^{\mathbb{N}} \middle| \exists \lim_{n\to\infty} r_n\right\}$$

- For every $a \in [-\infty, \infty]$, the set of sequences that converge to a:

$$\mathrm{ConvergeTo}\, a \coloneqq \left\{\vec{r} \in [-\infty,\infty]^{\mathbb{N}} \middle| \lim_{n\to\infty} r_n = a\right\}$$

- The set of convergence rates:

$$\mathrm{ConvergenceRate} \coloneqq \left\{\vec{r} \in (0,\infty] \middle| \lim_{n\to\infty} r_n = 0\right\}$$

◿

◸**5.2.** Show that the following higher-order operations are measurable:

- $\lim : \mathrm{Converge}[\infty,\infty] \to [-\infty,\infty]$
- $\liminf, \limsup : [-\infty,\infty]^{\mathbb{N}} \to [-\infty,\infty]$
- $\arg\min, \arg\max : [-\infty,\infty]^{\mathbb{N}} \to \mathbb{N}_\perp$
- $\min : \mathcal{B}_{\mathbb{N}} \setminus \{\varnothing\} \to \mathbb{N}$, where $\mathcal{B}_{\mathbb{N}}$ is the measurable space structure induced by identifying the measurable subsets of $\mathbb{N}$ with their indicator functions in the countable-product measurable space $2^{\mathbb{N}}$. ◿

◸**5.3.** For every measurable space X, we may adjoin a new element $\perp$ called 'bottom' representing the undefined value, and making the singleton $\{\perp\}$ measurable. Explicitly:

- The points are the disjoint union of the points in X and $\perp$: $\llcorner X_\perp \lrcorner \coloneqq \{\perp\} \amalg \llcorner X \lrcorner$.
- The measurable sets are generated by those of X and $\{\perp\}$:

$$\mathcal{B}_{X_\perp} \coloneqq \sigma(\{\{\iota_1 \perp\}\} \cup \iota_2 [[\mathcal{B}_X]])$$

We can use the undefined value to define partial measurable functions. Show that the following higher-order operations are measurable:

- $\lim : [-\infty,\infty]^{\mathbb{N}} \to [-\infty,\infty]_\perp$
- $\inf, \sup : ([-\infty,\infty]_\perp)^{\mathbb{N}} \to [-\infty,\infty]$
- $\mathrm{compress} : (X_\perp)^{\mathbb{N}} \to \left(X^{\mathbb{N}}\right)_\perp$ for any measurable space X which compresses the sequence by removing any intermediate undefined values. ◿

◸**5.4.** Define a measurable function $\mathrm{approx}\_ : \mathrm{ConvergenceRate} \times \mathbb{R} \to \mathbb{Q}^{\mathbb{N}}$, such that each $\mathrm{approx}_{\vec{b}} r$ is a sequence $\vec{q}$ of rational numbers that converges to r at rate $\vec{b}$, so for all $n \in \mathbb{N}$: $|q_n - r| < b_n$. ◿

6 Quasi-Borel spaces

Practice the basic definitions of qbses and their morphisms.

◸**6.1.** We can equip the real numbers with the structure of a qbs:

- The points are the real numbers.
- The random elements are the Borel measurable functions $\alpha : \mathbb{R} \to \mathbb{R}$

We'll write more succinctly below: $\mathbb{R} := \langle \llcorner \mathbb{R} \lrcorner, \mathbf{Meas}(\mathbb{R}, \mathbb{R}) \rangle$.

- Check that $\mathbb{R}$ satisfies the qbs axioms.
- Show that a function $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable iff it is a qbs morphism. ⊿

◸**6.2.** Let X be a set. The *indiscrete qbs over* X has all functions as random elements:

$$\underset{\mathbf{Qbs}}{\llcorner X \lrcorner} := \langle X, \mathbf{Set}(\mathbb{R}, X) \rangle$$

- Check that $\underset{\mathbf{Qbs}}{\llcorner X \lrcorner}$ satisfies the qbs axioms.
- Let A be any qbs. Show that every function $f : \underset{\mathbf{Set}}{\llcorner A \lrcorner} \to X$ is a qbs morphism:

$$f : A \to \underset{\mathbf{Qbs}}{\llcorner X \lrcorner}$$ ⊿

◸**6.3.** A *qbs structure* on a set X is a collection $\mathcal{R} \subseteq X^{\mathbb{R}}$ of functions closed under the qbs axioms. A function $\alpha : \mathbb{R} \to X$ is σ*-simple* when:

- The image $\alpha[\mathbb{R}]$ is countable; and
- For every $x \in \alpha[\mathbb{R}]$, the preimage $\alpha^{-1}[x] \subseteq \mathbb{R}$ is a Borel set.

Show the σ-simple functions are the smallest (w.r.t. set inclusion) qbs structure on X. ⊿

◸**6.4.** Let A, B, C be qbses. Show that the following functions are qbs morphisms:

- Constant functions: for every $b \in \underset{\mathbf{Set}}{\llcorner B \lrcorner}$, the function $(\lambda a.b) : A \to B$.
- Identity functions: $\mathrm{id} := (\lambda a.a) : A \to A$.
- If $f : B \to C$ and $g : A \to B$ are qbs morphisms then so is the composition $f \circ g : A \to C$.
- Every σ-simple functions $\alpha : \mathbb{R} \to A$. ⊿

◸**6.5.** Let X be a set. The *discrete qbs over* X has the σ-simple functions as random elements:

$$\overset{\mathbf{Qbs}}{\ulcorner X \urcorner} := \langle X, \{\alpha : \mathbb{R} \to X | \alpha \text{ is } \sigma\text{-simple}\} \rangle$$

By Ex.6.3, it is a qbs. Let A be any qbs. Show that every function $f : X \to \underset{\mathbf{Set}}{\llcorner A \lrcorner}$ is a qbs morphism:

$$f : \overset{\mathbf{Qbs}}{\ulcorner X \urcorner} \to A$$ ⊿

◸**6.6.** Let A, B be isomorphic qbses. Show that their sets of points and their sets of random elements are in bijection:

$$\llcorner A \lrcorner \cong \llcorner B \lrcorner \qquad \mathcal{R}_A \cong \mathcal{R}_B$$

(Recall from Ex.2.7 that two spaces A, B are isormorphic when there are two morphisms $f : A \to B$ and $g : B \to A$ that are each other's inverses: $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$.) ⊿

◸**6.7.** Show that the three spaces:

- $\mathbb{R}$, defined in Ex.6.1;
- $\llcorner \mathbb{R} \lrcorner_{\mathbf{Qbs}}$, defined in Ex.6.2; and
- $\ulcorner \mathbb{R} \urcorner^{\mathbf{Qbs}}$, defined in Ex.6.5

are pairwise non-isomorphic qbses. ⊿

◸**6.8.** Let $f : A \to B$ be a qbs morphism. Show:

- f is surjective iff f is an epimorphism in **Qbs**.
- f is injective iff f is a monomorphism in **Qbs**. ⊿

◸**6.9.** We have a functor $\llcorner - \lrcorner_{\mathbf{Set}} : \mathbf{Qbs} \to \mathbf{Set}$ sending each qbs A to its set of points.

- Define the action on morphisms, and show it is functorial and faithful.

Show:

- $\llcorner - \lrcorner_{\mathbf{Set}} : \mathbf{Qbs} \to \mathbf{Set}$ has both a left and a right adjoint. What are the unit, counit, and mate representations of each adjunction?
- The functor $\llcorner - \lrcorner_{\mathbf{Set}}$ is *essentially surjective*: every set is isomorphic to a set of points of some space.
- These left and right adjoints are fully-faithful, and neither essentially surjective. ⊿

7 Qbs constructions

In this sheet you'll construct new qbses out of given ones. If the development starts to feel too abstract, skip to the next sheet and come back to it when needed.

◸**7.1.** Let A be a qbs and $X \subseteq \llcorner A \lrcorner$ a set of points. We can equip X with a qbs structure by taking as random elements all the random elements of A whose image is in X:

$$\mathcal{R}_X := \{\alpha : \mathbb{R} \to X | \alpha \in \mathcal{R}_A\}$$

This qbs is called the *subspace of A induced by X*.

- Check that the subspace X is a qbs, and that the inclusion $X \hookrightarrow \llcorner A \lrcorner$ is a qbs morphism.

A *subspace embedding* $m : B \hookrightarrow A$ of a qbs B into A is a qbs morphism $m : B \to A$ where:

- $\llcorner m \lrcorner : \llcorner B \lrcorner \to \llcorner A \lrcorner$ is injective; and
- $\llcorner m \lrcorner \circ : \mathcal{R}_B \to \mathcal{R}_A$ is surjective.

Show the following:

- Every point $\underline{x} := \lambda \star .x : \mathbb{1} \to A$ is a subspace embedding.
- Each inclusion $X \subseteq \llcorner A \lrcorner$ of a subspace X into its superspace A is a subspace embedding.
- Not every monomorphism in **Qbs** is a subspace embedding.
- Every isomorphism is a subspace embedding.
- Every qbs morphism factors as the composition $m \circ e$ of an epimorphism followed by a subspace embedding. ◿

◸**7.2.** Let A be a qbs, X a set, and $g : X \to \llcorner A \lrcorner$ any function from X into the points of A. Show that g carries a qbs morphism $f : \ulcorner X \urcorner_{\mathbf{Qbs}} \to A$ from the indiscrete space over X into A, i.e., $\llcorner f \lrcorner = g$, iff the subspace $g[X] \hookrightarrow A$ is indiscrete.

Deduce that every morphism $f : \ulcorner X \urcorner_{\mathbf{Qbs}} \to \mathbb{R}$ is constant. ◿

◸**7.3.** Find the terminal qbs $\mathbb{1}$ and the initial qbs $\mathbb{0}$. ◿

◸**7.4.** Let X_1, X_2 be qbses. Construct their *product* $\langle X_1 \times X_2, \pi_1, \pi_2 \rangle$:

- The set of points is the cartesian product: $\llcorner X_1 \times X_2 \lrcorner := \llcorner X_1 \lrcorner \times \llcorner X_2 \lrcorner$.
- The random elements are tupling of random elements:

  $$\mathcal{R}_{X_1 \times X_2} := \{\alpha : \mathbb{R} \to X_1 \times X_2 | \pi_1 \circ \alpha \in \mathcal{R}_{X_1}, \pi_2 \circ \alpha \in \mathcal{R}_{X_2}\}$$

  We can think of random elements in $X_1 \times X_2$ as correlated random-elements in the product space.
- The two projections are given by the set-theoretic projections:

  $$\pi_i : X_1 \times X_2 \to X_i$$
  $$\pi_i \langle x_1, x_2 \rangle := x_i$$

Show:

- $X_1 \times X_2$ satisfies the qbs axioms.

- Each projection $\pi_i : X_1 \times X_2 \to X_i$ is a qbs morphism.
- The universal property of the product (see Ex.2.15).
- Generalise: construct the product $\prod_{i \in I} X_i$ of any I-indexed family of spaces. ◿

◸**7.5.** Let X_1, X_2 be qbses. Construct their *coproduct / disjoint union* $\langle X_1 \amalg X_2, \iota_1, \iota_2 \rangle$:

- The set of points is the disjoint union: $\llcorner X_1 \amalg X_2 \lrcorner := \llcorner X_1 \lrcorner \amalg \llcorner X_2 \lrcorner := (\{1\} \times X_1) \cup (\{2\} \times X_2)$.
- The random elements are binary recombinations of random elements:

$$\mathcal{R}_{X_1 \amalg X_2} := \left\{ \alpha : \mathbb{R} \to X_1 \amalg X_2 \,\middle|\, \begin{array}{l} \exists \alpha_1 \in \mathcal{R}_{X_1}, \alpha_2 \in \mathcal{R}_{X_2}. \alpha = [\alpha_1, \alpha_2] \\ \qquad := \lambda x. \begin{cases} x = \iota_1 x_1 : & \alpha_1\, x_1 \\ x = \iota_2 x_2 : & \alpha_2\, x_2 \end{cases} \end{array} \right\}$$

  We can think of random elements in $X_1 \amalg X_2$ as splitting the probability between the two spaces.
- The two injections are given by the set-theoretic injections:

$$\begin{aligned} &\iota_i : X_i \to X_1 \amalg X_2 \\ &\iota_i x := \iota_i x := \langle i, x \rangle \end{aligned}$$

Show:

- $X_1 \amalg X_2$ satisfies the qbs axioms.
- Each injection $\iota_i : X_i \to X_1 \amalg X_2$ is a qbs morphism.
- The universal property of the coproduct (see Ex.4.3).
- Generalise: construct the coproduct $\coprod_{i \in I} X_i$ of any I-indexed family of spaces. ◿

◸**7.6.** Let $X_1 \xrightarrow{f_1} Y \xleftarrow{f_2} X_2$ be two qbs morphisms. Construct their *pullback* $\left\langle f_1 \bowtie f_2, \overset{\bowtie}{\pi}_1, \overset{\bowtie}{\pi}_2 \right\rangle$:

- The set of points is the set-theoretic pullback:

$$\llcorner f_1 \bowtie f_2 \lrcorner := \left\{ \langle x_1, x_2 \rangle \in \llcorner X_1 \lrcorner \times \llcorner X_2 \lrcorner \,\middle|\, f_1\, x_1 = f_2\, x_2 \right\}$$

- The random elements are tupling of random elements whose projections agree:

$$\mathcal{R}_{f_1 \bowtie f_2} := \left\{ \alpha : \mathbb{R} \to \llcorner f_1 \bowtie f_2 \lrcorner \,\middle|\, \overset{\bowtie}{\pi}_i \circ \alpha \in \mathcal{R}_{X_i}, i = 1, 2 \right\}$$

- The projections are given by the set-theoretic projections:

$$\begin{aligned} &\overset{\bowtie}{\pi}_i : X_1 \times X_2 \to X_i \\ &\overset{\bowtie}{\pi}_i \langle x_1, x_2 \rangle := x_i \end{aligned}$$

Show:

- $f_1 \bowtie f_2$ satisfies the qbs axioms.
- Each projection $\overset{\bowtie}{\pi}_i : f_1 \bowtie f_2 \to X_i$ is a qbs morphism.
- The universal property of the pullback:

$$\begin{tikzcd}
f_1 \bowtie f_2 \arrow[r, "\overset{\bowtie}{\pi}_2"] \arrow[d, "\overset{\bowtie}{\pi}_1"'] \arrow[dr, phantom, "="] & X_2 \arrow[d, "f_2"] \\
X_1 \arrow[r, "f_1"'] & Y
\end{tikzcd}
\quad \text{and every} \quad
\begin{tikzcd}
Z \arrow[rr, bend left, "p_2"] \arrow[dr, "p_1"'] & & X_2 \arrow[d, "f_2"] \\
& X_1 \arrow[r, "f_1"'] & Y
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
Z \arrow[r, dashed, "h"] \arrow[rr, bend left, "p_2"] \arrow[dr, "p_1"'] & f_1 \bowtie f_2 \arrow[r, "\overset{\bowtie}{\pi}_2"'] \arrow[d, "\overset{\bowtie}{\pi}_1"] & X_2 \\
& X_1 &
\end{tikzcd}$$

- The pullback is a subspace of the product: $f_1 \bowtie f_2 \hookrightarrow X_1 \times X_2$. ⊿

◸**7.7.** The pullback of a subspace embedding $m : S \hookrightarrow Y$ along any morphism $f : X \hookrightarrow Y$ is a subspace embedding $\overset{\bowtie}{\pi}_1 : f \bowtie m \hookrightarrow X$. ⊿

We can summarise this situation in the previous exercise in this diagram, where the right-angle marker inside the square marks it as a pullback:

$$\begin{tikzcd}
f_1 \bowtie f_2 \arrow[r, "\overset{\bowtie}{\pi}_2"] \arrow[d, hook, "\overset{\bowtie}{\pi}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, hook, "m"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

◸**7.8.** Show:

- The free qbs functor $\ulcorner - \urcorner^{\mathbf{Qbs}} : \mathbf{Set} \to \mathbf{Qbs}$ preserves finite products.
- The free qbs functor $\ulcorner - \urcorner^{\mathbf{Qbs}} : \mathbf{Set} \to \mathbf{Qbs}$ doesn't preserve products.
- The indiscrete qbs functor $\llcorner - \lrcorner_{\mathbf{Qbs}} : \mathbf{Set} \to \mathbf{Qbs}$ doesn't preserve finite coproducts.
- Deduce that the sequence of adjunctions: $\ulcorner - \urcorner^{\mathbf{Qbs}} \dashv \llcorner - \lrcorner_{\mathbf{Set}} \dashv \llcorner - \lrcorner_{\mathbf{Qbs}}$ between **Qbs** and **Set** doesn't have a further left adjoint nor a further right adjoint. ⊿

Let X be a set. A *metaphorology*[1] over X is a set $\mathcal{R} \subseteq X^{\mathbb{R}}$ of functions from $\mathbb{R}$ to X that satisfies the qbs axioms (contains all constant functions and closed and measurable pre-composition and recombination). Thus, a qbs A is a set $\llcorner A \lrcorner$ equipped with a metaphorology $\mathcal{R}_A$. (In Ex.6.3 we called this concept a qbs structure.)

◸**7.9.** Let X be a set and $E \subseteq X^{\mathbb{R}}$ any set of functions. Show that the smallest metaphorology $\mathrm{Cl}_{\mathrm{qbs}}E$ on X containing E is given by the recombinations of measurable pre-compositions of E-elements and constant functions:

$$\mathrm{Cl}_{\mathrm{qbs}}E = \left\{ [\lambda r \in U_i.\alpha_i(r)]_{i \in I} \;\middle|\; \begin{array}{l} I \text{ is countable}, \mathbb{R} = \biguplus_{i \in I} U_i, \text{ and for every } i \in I\text{:} \\ U_i \in \mathcal{B}_{\mathbb{R}}, \text{and} \\ \text{either } \alpha_i \text{ constant, or there is some} \\ \text{measurable } \varphi_i : U_i \to \mathbb{R},\ \beta_i \in E \text{ s.t.: } \alpha_i = \beta_i \circ \varphi_i \end{array} \right\}$$ ⊿

◸**7.10.** Show that the functor $\llcorner - \lrcorner_{\mathbf{Set}} : \mathbf{Qbs} \to \mathbf{Set}$ generates limits and colimits, but doesn't create limits nor colimits (see discussion before Ex.3.27). Deduce that **Qbs** is complete and cocomplete. ⊿

[1] I'm open to suggestions for other names. Going back to its original roots, 'metaphor' originates from the Greek $\mu\varepsilon\tau\alpha$ ('meta', across) and $\varphi\varepsilon\rho\omega$ ('phero', to carry). This choice makes 'metaphors' an appealing alternative to 'random element'. The other candidate was 'stochastology'.

8 Borel subspaces

The central notion in measure theory is that of a measurable subset — it is the defining concept of a measurable space. With quasi-Borel spaces, measurable subsets are a derived notion, but take a nonetheless central role.

◸**8.1.** A *measurable*, or *Borel*, subset in a qbs A is a subset $U \subseteq \llcorner A \lrcorner$ such that the preimage under every random element $\alpha \in \mathcal{R}_A$ is a Borel subset of the reals: $\alpha^{-1}[U] \in \mathcal{B}$. We denote by $\mathcal{B}_A$ the set of Borel subsets of A.

- Show that the measurable sets $\mathcal{B}_A$ in a qbs A form a σ-algebra, and every random element is measurable w.r.t. this σ-algebra.
  We denote the resulting measurable space by $\overset{\mathbf{Meas}}{\ulcorner A \urcorner} := \left\langle \underset{\mathbf{Set}}{\llcorner A \lrcorner}, \mathcal{B}_A \right\rangle$, and call it the *free* measurable space over A.
- Show that $U \subseteq \llcorner A \lrcorner$ is measurable iff its indicator function $[- \in U] : A \to \overset{\mathbf{Qbs}}{\ulcorner \mathbb{2} \urcorner}$ is a qbs morphism from A into the discrete qbs on the two-element set. ◿

◸**8.2.** Find the Borel sets of the discrete qbs $\overset{\mathbf{Qbs}}{\ulcorner \mathbb{2} \urcorner}$ and the indiscrete qbs $\underset{\mathbf{Qbs}}{\llcorner \mathbb{2} \lrcorner}$ on two elements. Generalise this result to the discrete and indiscrete qbses over any set X. ◿

◸**8.3.** Show that the Borel subsets of $\mathbb{R}$ in the standard sense coincide with the measurable subsets of the qbs $\mathbb{R}$. ◿

◸**8.4.** Let A be a qbs and $X \subseteq \llcorner A \lrcorner$ be a subset.

- Show that if $U \subseteq \llcorner A \lrcorner$ is Borel in A, then $U \cap X$ is Borel in the subspace X:

$$U \in \mathcal{B}_A \implies U \cap X \in \mathcal{B}_X$$

- Show that if X is itself a Borel subset, then $\mathcal{B}_X \subseteq \mathcal{B}_A$.
- Show that the previous clause may fail if X is not Borel. ◿

The Borel subsets of a subspace can be quite different from the Borel subsets of its superspace. For example, we may have a Borel subset $V \in \mathcal{B}_X$ of the subspace that is not of the form $U \cap X$ for any Borel subset $U \in \mathcal{B}_A$ of the superspace.
Here's the intuition:

- A subset U in a qbs is measurable unless there is some random element that stops it from being measurable by mapping U onto a non-Borel inverse image.
- 'Wild' random elements may not factor through a subspace embedding $X \hookrightarrow A$.
- So a subspace may have more Borel subsets in X than in its superspace.

If you want to see this intuition playing out, here is how to construct a counter-example:

◸**8.5.** Let $C_1 \subseteq \mathbb{R}$ be a non-Borel subset and $C_2 := \mathbb{R} \setminus C_1$ its complement, also non-Borel. Let $\mathbb{3} := \{0, 1, 2\}$ be a three-element set, and define two primitive random elements $\alpha_i : \mathbb{R} \to \mathbb{3}$:

$$\alpha_0 r := \begin{cases} r \in C_1 : 0 \\ r \in C_2 : 2 \end{cases} \qquad \alpha_1 r := \begin{cases} r \in C_1 : 1 \\ r \in C_2 : 2 \end{cases}$$

Take $A := \langle \mathbb{3}, \mathrm{Cl}_{\mathrm{qbs}} \{\alpha_0, \alpha_1\} \rangle$ to be the qbs over $\mathbb{3}$ with the smallest metaphorology (see Ex.7.9) containing α_0 and α_1, and take $X := \mathbb{2} \subseteq \mathbb{3}$.

- Show that $X, \{0\}, \{0,2\} \notin \mathcal{B}_A$ are not Borel subsets in A.
- Show that if $\alpha \in \mathcal{R}_A$ is a random element in A, then either α is σ-simple or $2 \in \mathrm{Im}\,(\alpha)$.
- Show that $\{0\} \in \mathcal{B}_X$ is a Borel subset of the subspace X. ◿

◸**8.6.** Let $f : A \to B$ be a qbs morphism. Show that:

- The inverse image under f restricts to a function $\mathcal{B}_f : \mathcal{B}_B \to \mathcal{B}_A$.
- The underlying function $\llcorner f \lrcorner_{\mathbf{Set}}$ is a measurable function $\ulcorner f \urcorner^{\mathbf{Meas}} : \ulcorner A \urcorner^{\mathbf{Meas}} \to \ulcorner B \urcorner^{\mathbf{Meas}}$. ◿

The collection of Borel sets has a universal property: it allows us to connect measurable spaces with quasi-Borel spaces as follows:

◸**8.7.** For a measurable space M, define its set of *random elements* by $\mathcal{R}_M := \mathbf{Meas}(\mathbb{R}, M)$.

- Show that $\mathcal{R}_M$ is a metaphorology, that is, $\llcorner M \lrcorner_{\mathbf{Qbs}} := \left\langle \llcorner M \lrcorner_{\mathbf{Set}}, \mathcal{R}_M \right\rangle$ is a qbs.
- For every measurable function $f : M \to N$ between measurable spaces, show that its underlying function is a qbs morphism $\llcorner f \lrcorner_{\mathbf{Qbs}} : \llcorner M \lrcorner_{\mathbf{Qbs}} \to \llcorner N \lrcorner_{\mathbf{Qbs}}$.
- Noticing that $\llcorner - \lrcorner_{\mathbf{Qbs}} : \mathbf{Meas} \to \mathbf{Qbs}$ is a (faithful) functor, show that it has a left adjoint equipping a qbs with its set of Borel subsets: $\ulcorner - \urcorner^{\mathbf{Meas}} \dashv \llcorner - \lrcorner_{\mathbf{Qbs}}$. ◿

◸**8.8.** The free qbs functor $\ulcorner - \urcorner^{\mathbf{Qbs}} : \mathbf{Set} \to \mathbf{Qbs}$ doesn't preserve countable products. ◿

This point is a natural place to stop, but if you're having fun with this material, then the rest of this sheet studies the relationships between natural notions of 'subspace'.

- $m : A \rightarrowtail B$ Monomorphisms: injective qbs morphisms.
- $m : A \hookrightarrow B$ Subspace embedding: injective on elements and surjective on random-elements that factor through the image.
- $m : A \rightarrowtail B$ Borel injections: monomorphisms whose image is a Borel subset.
- $m : A \hookrightarrow B$ Borel embeddings: subspace embeddings whose image is a Borel subset.

We establish their following mutual relationships, where all inclusions are proper:

$$
\begin{tikzcd}
 & \textbf{mono} & \\
 & & \textbf{subspace embedding} \arrow[ul, hook] \arrow[d, equal] \\
\textbf{Borel mono} \arrow[uur, hook] & & \textbf{strong mono} \arrow[d, equal] \\
 & & \textbf{regular mono} \\
 & \textbf{Borel embedding} \arrow[uul, hook] \arrow[ur, hook] &
\end{tikzcd}
$$

◸**8.9.** Place the following injections in the hierarchy of monomorphisms above:

- The injection $\top := \lambda \star.1 : \ulcorner \mathbb{1} \urcorner^{\mathbf{Qbs}} \rightarrowtail \ulcorner \mathbb{2} \urcorner^{\mathbf{Qbs}}$.
- The injection $\lambda x.x : \ulcorner \mathbb{2} \urcorner^{\mathbf{Qbs}} \rightarrowtail \llcorner \mathbb{2} \lrcorner_{\mathbf{Qbs}}$.

- The injection $\lambda x.x : \ulcorner \mathbb{2} \urcorner^{\mathbf{Qbs}} \rightarrowtail \llcorner \mathbb{3} \lrcorner_{\mathbf{Qbs}}$.
- The (subspace) inclusion $\lambda x.x : C \hookrightarrow \mathbb{R}$ where C is a non-Borel subset of $\mathbb{R}$. ⊿

◸**8.10.** Let $m : S \to A$ be a qbs morphism. Show that the following are equivalent:

- m is a subspace embedding, i.e.: there is a subset $X \subseteq \llcorner A \lrcorner$ and an isomorphism $m' : B \xrightarrow{\cong} X$ satisfying:

$$\begin{tikzcd} S \arrow[dr, hook, "m"] \arrow[d, "m'"', "\cong"] & \\ X \arrow[r, phantom, "="] \arrow[r, hook, "\lambda x.x"'] & A \end{tikzcd}$$

- m is *right-orthogonal* to every empimorphism $e : B \twoheadrightarrow C$: for every commuting square as on the left, there is a unique morphism $h : C \to S$ commuting the triangles on the right:

$$\begin{tikzcd} B \arrow[r, two heads, "e"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & C \arrow[d, "g"] \\ S \arrow[r, hook, "m"'] & A \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} B \arrow[r, two heads, "e"] \arrow[d, "f"'] & C \arrow[d, "g"] \arrow[dl, dashed, "h"'] \\ S \arrow[r, hook, "m"'] & A \end{tikzcd}$$

(Morphisms that have this property are called *strong monomorphisms*.)

- m is an *equaliser* of some parallel pair of morphisms $f, g : A \to B$:
  - m *equalises* f and g:

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ S \arrow[ur, "m"] \arrow[dr, "m"'] & = & B \\ & A \arrow[ur, "g"'] & \end{tikzcd}$$

  - and every equalising morphism $e : C \to A$ factors uniquely through m:

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ E \arrow[ur, bend left, "e"] \arrow[dr, bend right, "e"'] & = & B \\ & A \arrow[ur, "g"'] & \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} & & A \\ E \arrow[urr, bend left, "e"] \arrow[r, dashed, "h"'] & E \arrow[ur, "m"'] & \end{tikzcd}$$

  (Morphisms that have this property are called *regular monomorphisms*.) ⊿

◸**8.11.** A class of qbs-morphisms is *admissible* when, for every pullback square as follows, in which $m \in \mathcal{M}$ then necessarily $\overset{\bowtie}{\pi}_1 \in \mathcal{M}$:

$$\begin{tikzcd} f \bowtie m \arrow[r, "\overset{\bowtie}{\pi}_2"] \arrow[d, "\overset{\bowtie}{\pi}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "m"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Show that:

- Monomorphisms are admissible.
- Subspace embeddings are admissible.
- Borel embeddings are admissible. ⊿

◸**8.12.** Let $\mathcal{M}$ be an admissible class. An $\mathcal{M}$-*classifier* is a pair $\langle \Omega_{\mathcal{M}}, \top_{\mathcal{M}} \rangle$ consisting of:

- a space $\Omega_{\mathcal{M}}$; and
- an $\mathcal{M}$-morphism $\top_{\mathcal{M}} : \mathbb{1} \to \Omega_{\mathcal{M}}$

such that for every $\mathcal{M}$-morphism $m : X \to A$, there is a unique qbs morphism $\varphi : A \to \Omega_{\mathcal{M}}$ for which the following square is a pullback square:

$$\begin{tikzcd} X \arrow[r, "\langle\rangle"] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{1} \arrow[d, "\top_{\mathcal{M}}"] \\ A \arrow[r, dashed, "\varphi"'] & \Omega_{\mathcal{M}} \end{tikzcd}$$

In this case, we denote this unique φ by $[- \in m[X]]_{\mathcal{M}} : A \to \Omega_{\mathcal{M}}$.
Show:

- If $\mathcal{M}$ has a classifier in **Qbs**, then $\mathcal{M}$ contains only subspace embeddings.
- The indiscrete Booleans $\langle \llcorner \mathbb{2} \lrcorner_{\mathbf{Qbs}}, \underline{\mathbf{true}} \rangle$ form a subspace embedding classifier.
- The discrete Booleans $\langle \ulcorner \mathbb{2} \urcorner^{\mathbf{Qbs}}, \underline{\mathbf{true}} \rangle$ form a Borel embedding classifier.
- There are no monomorphism nor Borel monomorphism classifiers in **Qbs**. ⊿

A *factorisation system* $\langle \mathcal{E}, \mathcal{M} \rangle$ is a pair of classes of morphisms such that:

- $\mathcal{E}$ and $\mathcal{M}$ are closed under composition and contain all isomorphisms;
- every morphism $f : A \to B$ has an $\mathcal{E}$-$\mathcal{M}$ *factorisation*:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\mathcal{E} \ni e"'] & {} \arrow[d, phantom, "="] & B \\ & X \arrow[ur, "m \in \mathcal{M}"'] & \end{tikzcd}$$

- every morphism $m \in \mathcal{M}$ is right-orthogonal to every morphism $e \in \mathcal{E}$ (cf. Ex.8.10):

$$\begin{tikzcd} B \arrow[r, "e \in \mathcal{E}"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & C \arrow[d, "g"] \\ S \arrow[r, "m \in \mathcal{M}"'] & A \end{tikzcd} \implies \begin{tikzcd} B \arrow[r, "e \in \mathcal{E}"] \arrow[d, "f"'] & C \arrow[d, "g"] \arrow[dl, dashed, "h"'] \\ S \arrow[r, "m \in \mathcal{M}"'] & A \end{tikzcd}$$

◸**8.13.** Show that $\langle \textbf{epi}, \textbf{subspace embedding} \rangle$ is a factorisation system. ⊿

◸**8.14.** A qbs morphism $e : A \to B$ is a *strong epimorphism* when the its action on random elements is surjective:

$$e \circ - : \mathcal{R}_A \twoheadrightarrow \mathcal{R}_B$$

Show that:

- The projection $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ is a strong epimorphism.
- Every strong epimorphism is surjective.
- Every map from a non-empty space into the terminal space $\langle\rangle : X \to \mathbb{1}$ is a strong epimorphism.

- If $f_i : A_i \to B_i$, $i \in I$, is a countable collection of strong epimorphisms, then their product $\prod_{i \in I} f_i : \prod_{i \in I} A_i \to \prod_{i \in I} B_i$ is a strong epimorphism. ◿

◸**8.15.** Find an epimorphism that is not a strong epimorphism. ◿

◸**8.16.** Show that $\langle \mathbf{strong\ epimorphisms}, \mathbf{mono} \rangle$ is a factorisation system. ◿

9 Function spaces

Let A, B be qbses, their *function* space B^A is given by the following:

- The set of points $\llcorner B^A \lrcorner$ is the set $\mathbf{Qbs}(A, B)$ of qbs morphisms $f : A \to B$.
- The set of random elements $\mathcal{R}_{B^A}$ is the set $\mathrm{curry}\,[\mathbf{Qbs}(\mathbb{R} \times A, B)]$, consisting of functions $\alpha : \mathbb{R} \to \mathbf{Qbs}(A, B)$ that, when uncurried, are qbs morphisms $\mathrm{uncurry}(\alpha) : \mathbb{R} \times A \to B$.

Exercises Ex.9.1–Ex.9.3 unpack these definitions and show that they realise the familiar interface to functions — evaluation/application and abstraction — as well as the qbs axioms. You can skip them and come back after you've used the function space in the later exercises.

◸**9.1.** Let A, B be qbses and $\gamma : \mathbb{R} \times \llcorner A \lrcorner \to \llcorner B \lrcorner$ be any set theoretic function. Show:

- If $\gamma : \mathbb{R} \times A \to B$ is a qbs morphism, then its curried form $\mathrm{curry}\,\gamma := \lambda r.\lambda a.\gamma(r, a)$ is pointwise a qbs morphism: $\mathrm{curry}\,\gamma : \mathbb{R} \to \mathbf{Qbs}(A, B)$.
- $\mathrm{curry}\,\gamma \in \mathcal{R}_{B^A}$ iff for every measurable $\varphi : \mathbb{R} \to \mathbb{R}$ and random element $\alpha \in \mathcal{R}_A$, the following function is a random element:

$$(\lambda r.\gamma(\varphi\, r, \alpha\, r)) \in \mathcal{R}_B$$ ◿

◸**9.2.** Let A, B be qbses.

- Validate the constant and precomposition qbs axioms for the function space.
- Let $\vec{\gamma} \in \mathcal{R}_{B^A}$ be a sequence of random elements in the function space, $\mathbb{R} = \biguplus_n U_n$ a countable partition of the reals into Borel sets, $\varphi : \mathbb{R} \to \mathbb{R}$ a measurable function, and $\alpha \in \mathcal{R}_A$ a random element. Let $V_n := \varphi^{-1}[U_n]$.
  - Evaluate the recombination $\langle \lambda r.(\mathrm{uncurry}\,\gamma_n)(\varphi\, r, \alpha\, r)\rangle_n$ along $\vec{V}$ at any $s \in V_m$.
  - Let γ be the recombination of $\vec{\gamma}$ along $\vec{U}$. Evaluate $(\mathrm{uncurry}\,\gamma)(\varphi\, s, \alpha\, s)$ at any $s \in V_m$.
  - Validate the recombination axiom for the function space.
- Show that evaluation $\mathrm{eval} : B^A \times A \to A$ is a qbs morphism. ◿

◸**9.3.** Show that $\left\langle B^A, \mathrm{eval}\right\rangle$ is the exponential of B by A (cf. Sec. 4). ◿

We equip the Borel subsets $\mathcal{B}_A$ of a qbs A with the structure of a qbs. Identifying a Borel subset $U \subseteq \llcorner A \lrcorner$ with its indicator function $[- \in U] : A \to \mathbb{2}$ (cf. Ex.8.1), we *define*

$$\mathcal{B}_A := \mathbb{2}^A$$

◸**9.4.** Show that the following functions are qbs morphisms:

- Membership testing: $(\in) : A \times \mathcal{B}_A \to \mathbb{2}$
- Complementation: $\neg : \mathcal{B}_A \to \mathcal{B}_A$
- Countable unions and intersection: $\bigcup_I, \bigcap_I : \mathcal{B}_A^I \to \mathcal{B}_A$, for I countable set. ◿

◸**9.5.** Let A be a qbs with a countable carrier. Show that the following functions are qbs morphisms:

- Equality testing, subset containment: $(=), (\subseteq), (\subset) : \mathcal{B}_A^2 \to \mathcal{B}_A$.
- Inhabitation: $(\neq \varnothing) : \mathcal{B}_A \to \mathbb{2}$. ◿

◸**9.6.** Using the techniques in Sec. A, there is a Borel set $U \in \mathcal{B}_{\mathbb{R}}$ and a measurable function $f : \mathbb{R} \to \mathbb{R}$ such that the image $f[\mathbb{R}]$ is not Borel (cf. Ex.A.13).
Use this fact and show that the following functions are *not* qbs morphisms:

- Inhabitation: $(\neq \varnothing) : \mathcal{B}_{\mathbb{R}} \to \mathbb{2}$.
- Equality and containment $(=), (\subseteq), (\subset) : \mathcal{B}_{\mathbb{R}}^2 \to \mathcal{B}_{\mathbb{R}}$.
- Disjointness: $(- \cap - = \varnothing) : \mathcal{B}_{\mathbb{R}}^2 \to \mathbb{2}$. ◿

◸**9.7.** If $A = \ulcorner \overset{\mathbf{Qbs}}{I} \urcorner$ is a *finite* discrete qbs, then $\mathcal{B}_A$ is a finite discrete qbs. ◿

◸**9.8.** Let $U \subseteq \mathcal{B}_{\mathbb{R}\times\mathbb{R}}$ be a Borel set. Its *section* at r is the set $U_r \coloneqq \{s \in \mathbb{R} | \langle r, s\rangle \in U\}$. A set $\mathcal{U} \subseteq \mathcal{B}_{\mathbb{R}}$ is *Borel on Borel* when, for every Borel set $U \subseteq \mathbb{R} \times \mathbb{R}$, the set of sections of U that are in $\mathcal{U}$ is Borel: $\{r \in \mathbb{R} | U_r \in \mathcal{U}\} \in \mathcal{B}_{\mathbb{R}}$.
Show that $\mathcal{U}$ is Borel on Borel iff $\mathcal{U} \in \mathcal{B}_{\mathcal{B}_{\mathbb{R}}}$.
The observation that this descriptive-set-theoretic notion coincides with the Borel sets on a higher-order space is due to Sabok et al. (2021). ◿

◸**9.9.** Let A be a qbs. Show that a function $\alpha : \mathbb{R} \to \llcorner A \lrcorner$ is a random element in $\mathcal{R}_A$ iff it is a qbs morphism $\alpha : \mathbb{R} \to A$. ◿

The last exercise provides a qbs of random elements, by setting: $\mathcal{R}_A \coloneqq A^{\mathbb{R}}$.

◸**9.10.** Define functors $\mathcal{B}_- : \mathbf{Qbs}^{\mathrm{op}} \to \mathbf{Qbs}$ and $\mathcal{R}_- : \mathbf{Qbs} \to \mathbf{Qbs}$, and construct a natural isomorphism $\mathcal{R}_{\mathcal{B}_A} \cong \mathcal{B}_{\mathbb{R}\times A}$. ◿

◸**9.11.** Show that $\mathcal{R}_-$ preserves strong epimorphisms (cf. Ex.8.14): if $e : A \twoheadrightarrow B$ is a strong epimorphism, then $\mathcal{R}_e : \mathcal{R}_A \to \mathcal{R}_B$ is also a strong epimorphism. ◿

10 Type structure

The type combinators — tuples, variants, functions, and their recursive combinations — are the basic building-blocks of compositional programming, and we can similarly use them as building-blocks for statistical modelling. This sheet covers classical material in the semantics of type structure, specialised for quasi-Borel spaces.

We've already seen the qbs constructions that do much of the low-level work: the *ground* combinators — products and coproducts.
Given a *sequence* of spaces spaces $A_1, \ldots, A_n$, their:

- *tuple* space of is the product $A_1 \times \cdots \times A_n$ with elements the n-tuples $\langle a_1, \ldots, a_n \rangle$ (Ex.7.4).
- *variant* space is the coproduct $A_1 \amalg \cdots \amalg A_n$ with elements $\iota_i a$, $1 \le i \le n$, $a \in \llcorner A_i \lrcorner$ (Ex.7.5).

When modelling, as with programming, using positional tuples and variants can be tedious and confusing, and doesn't scale well to large or structured collections of spaces. So we also introduce the indexed versions of tuples, called *records*, and indexed variants. Given an I-indexed *set* of spaces $\langle A_i \rangle_{i \in I}$, their:

- *record* space is the I-indexed product $\prod_{i \in I} A_i$.
- *variant* space is the I-indexed coproduct $\coprod_{i \in I} A_i$.

The structure of these spaces depends on a set rather than a sequence, and so only the labels matter, not their order. We'll therefore use set-comprehension-like notation for the indexed versions, to emphasise that the order of components doesn't matter, only their indices. So when $I = \{\ell_1, \ldots, \ell_n\}$, we'll use:

- $\langle \ell_1 : A_{\ell_1}, \ldots, \ell_n : A_{\ell_n} \rangle \coloneqq \prod_{\ell \in I} A_\ell$
  with element records $\langle \ell_1 : a_{\ell_1}, \ldots, \ell_n : a_{\ell_n} \rangle \coloneqq \langle a_\ell \rangle_{\ell \in I}$.
- $\{\ell_1 : A_{\ell_1} \mid \ldots \mid \ell_n : A_{\ell_n}\} \coloneqq \coprod_{\ell \in I} A_\ell$ with 'constructor-headed' elements $\ell\, a \coloneqq \iota_\ell a$ for $\ell \in I$.

A The Borel hierarchy

These exercises concern the details behind the proof of Aumann's (1961) theorem. Flicking through, you'll see there's quite a lot to cover, but the rest of the material doesn't depend on this technical development. It's only here to satisfy your curiosity about what happens deep inside the σ-algebra of Borel sets. If you enjoy these, take a closer look at *descriptive set theory*. Two classical textbooks are Moschovakis's (1987) selection of key, central results, and Kechris's (1995) comprehensive, detailed, and slightly more modern book.
Define by transfinite induction on $\omega_1 + 1$, the successor of the first uncountable ordinal:

$$\begin{aligned}
&\mathbf{\Sigma}^{\mathcal{U}}_{\alpha}, \mathbf{\Pi}^{\mathcal{U}}_{\alpha}, \mathbf{\Delta}^{\mathcal{U}}_{\alpha} \subseteq \wp X && (\alpha \in \omega_1)\\
&\mathbf{\Sigma}^{\mathcal{U}}_{1} := \mathcal{U} \\
&\mathbf{\Sigma}^{\mathcal{U}}_{\alpha+1} := \left\{ \bigcup_{i\in I} A_i \,\middle|\, I \subseteq \mathbb{N}, \vec{A} \in \mathcal{U} \cup \bigcup_{\beta\le\alpha} \mathbf{\Pi}^{\mathcal{U}}_{\beta} \right\} && (1 \le \alpha \in \omega_1)\\
&\mathbf{\Sigma}^{\mathcal{U}}_{\gamma} := \bigcup_{\beta<\gamma} \mathbf{\Sigma}^{\mathcal{U}}_{\beta} && (1 \le \gamma \text{ a limit ordinal in } \omega_1)\\
&\mathbf{\Pi}^{\mathcal{U}}_{\alpha} := \left[\mathbf{\Sigma}^{\mathcal{U}}_{\alpha}\right]^{\mathsf{C}} := \left\{ A^{\mathsf{C}} \middle| A \in \mathbf{\Sigma}^{\mathcal{U}}_{\alpha} \right\} \qquad \mathbf{\Delta}^{\mathcal{U}}_{\alpha} := \mathbf{\Sigma}^{\mathcal{U}}_{\alpha} \cap \mathbf{\Delta}^{\mathcal{U}}_{\alpha}
\end{aligned}$$

◸**A.1.** For every $\alpha \le \omega_1$, we have $\mathbf{\Sigma}^{\mathcal{U}}_{\alpha} \cup \mathbf{\Pi}^{\mathcal{U}}_{\alpha} \subseteq \mathbf{\Delta}^{\mathcal{U}}_{\alpha+1}$. ◿

◸**A.2.** Prove that $\sigma(\mathcal{U}) = \mathbf{\Sigma}^{\mathcal{U}}_{\omega_1} = \mathbf{\Pi}^{\mathcal{U}}_{\omega_1} = \mathbf{\Delta}^{\mathcal{U}}_{\omega_1}$. ◿

We therefore have the following relationships between the classes of the *Borel hierarchy*:

$$\begin{tikzcd}[column sep=small]
 & \mathbf{\Sigma}^{\mathcal{U}}_{1} \arrow[dr, hook] & & \mathbf{\Sigma}^{\mathcal{U}}_{2} \arrow[dr, hook] & & \mathbf{\Sigma}^{\mathcal{U}}_{3} \arrow[dr, hook] & & & \mathbf{\Sigma}^{\mathcal{U}}_{\omega} \arrow[dr, hook] & & \mathbf{\Sigma}^{\mathcal{U}}_{\omega+1} \arrow[dr, hook] & & & \mathbf{\Sigma}^{\mathcal{U}}_{\omega_1} \arrow[dr, equal] & \\
\mathbf{\Delta}^{\mathcal{U}}_{1} \arrow[ur, hook] \arrow[dr, hook] & & \mathbf{\Delta}^{\mathcal{U}}_{2} \arrow[ur, hook] \arrow[dr, hook] & & \mathbf{\Delta}^{\mathcal{U}}_{3} \arrow[ur, hook] \arrow[dr, hook] & & \cdots \subseteq & \mathbf{\Delta}^{\mathcal{U}}_{\omega} \arrow[ur, hook] \arrow[dr, hook] & & \mathbf{\Delta}^{\mathcal{U}}_{\omega+1} \arrow[ur, hook] \arrow[dr, hook] & & \cdots \subseteq & \mathbf{\Delta}^{\mathcal{U}}_{\omega_1} \arrow[ur, equal] \arrow[dr, equal] & & \sigma(\mathcal{U}) \\
 & \mathbf{\Pi}^{\mathcal{U}}_{1} \arrow[ur, hook] & & \mathbf{\Pi}^{\mathcal{U}}_{2} \arrow[ur, hook] & & \mathbf{\Pi}^{\mathcal{U}}_{3} \arrow[ur, hook] & & & \mathbf{\Pi}^{\mathcal{U}}_{\omega} \arrow[ur, hook] & & \mathbf{\Pi}^{\mathcal{U}}_{\omega+1} \arrow[ur, hook] & & & \mathbf{\Pi}^{\mathcal{U}}_{\omega_1} \arrow[ur, equal] &
\end{tikzcd}$$

Given a set V whose elements represent variables, the σ*-terms* over V are the countably-infinitary terms generated by the following grammar:

$$t, s ::= x \mid x^{\mathsf{C}} \mid \bigcup_{i\in I} t_i \mid \bigcap_{i\in I} t_i \qquad (x \in V, I \subseteq \mathbb{N})$$

Given a *valuation* $e : V \to \sigma(\mathcal{U})$, we can interpret each σ-term t as a Borel subset $[\![t]\!]\, e \in \sigma(\mathcal{U})$. Note that every term t involves only countably many variables, we call these variables its *support* suppt.

◸**A.3.** Let $\mathcal{U} \subseteq \wp X$, $\mathcal{V} \subseteq \wp Y$. Show that for every measurable $f : \langle X, \sigma(\mathcal{U})\rangle \to \langle Y, \sigma(\mathcal{V})\rangle$, the inverse image f^{-1} is a homomorphism of σ-terms:

$$f^{-1}[[\![t]\!]\, e] = [\![t]\!]\,(f^{-1} \circ e)$$ ◿

◸**A.4.** Show that if $e : V \twoheadrightarrow \mathcal{U}$ is surjective, then $[\![-]\!]\, e$ is surjective on $\sigma(\mathcal{U})$. ◿

We call a term *alternating* when, for every non-variable sub-term $f\,\langle t_i\rangle_i$, the root of each direct sub-tree is not the same operation symbol f.

◸**A.5.** Show that every term is denotationally equivalent to an alternating term. You might enjoy presenting a denotation-preserving terminating rewriting system. ◿

The *Aumann rank* function assigns to each Borel set the first stage in the hierarchy in which it occurs in some $\mathbf{\Sigma}^{\mathcal{U}}$ set:

$$\mathrm{rank}^{\mathcal{U}} : \sigma(\mathcal{U}) \to \omega_1$$
$$\mathrm{rank}^{\mathcal{U}} A := \min\left\{\alpha \in \omega_1 \middle| A \in \mathbf{\Sigma}^{\mathcal{U}}_{\alpha}\right\}$$

Define the *alternating depth* of a σ-term as follows:

$$\mathrm{alter} : \sigma\text{-Term}V \to \omega_1$$
$$\mathrm{alter}\, x := \mathrm{alter}\, x^{\mathsf{C}} := 0 \quad \mathrm{alter} \bigcup_{i\in I} t_i := \bigvee_{i \in I} \mathrm{alter}\, t_i \quad \mathrm{alter} \bigcap_{i\in I} t_i := \bigvee_{i\in I} \mathrm{alter}\, t_i + \bigvee_{i\in I}\left[t_i \neq \bigcap_{j\in J} s_j\right]$$

◸**A.6.** Let t be a σ-term and e a valuation in some $\mathcal{U}$.

- Show that $[\![t]\!]\, e \in \mathbf{\Sigma}^{\mathcal{U}}_{\mathrm{alter}\, t \vee \alpha}$, where $\alpha := \bigvee_{x \in \mathrm{suppt}}\, e(x) \in \omega_1$.
- Deduce that if $e : V \to \mathcal{U}$, then $\mathrm{rank}\, [\![t]\!]\, e \le \mathrm{alter}\, t$. Generalise to any $e : V \to \sigma(\mathcal{U})$.
- Show that $\mathrm{rank} A = \min\{\mathrm{alter}\, t | A = [\![t]\!]\, e\}$. ◿

◸**A.7.** Prove that if $A \in \sigma(\mathcal{U})$ and $\rho := \mathrm{rank}^{\mathcal{U}} A$, then:

$$A \cap \left[\mathbf{\Sigma}^{\mathcal{U}}_{\alpha}\right] \subseteq \mathbf{\Sigma}^{A\cap[\mathcal{U}]}_{\alpha} \subseteq \mathbf{\Sigma}^{\mathcal{U}}_{(\rho+1)\vee\alpha} \quad A \cap \left[\mathbf{\Pi}^{\mathcal{U}}_{\alpha}\right]$$
$$\subseteq \mathbf{\Pi}^{A\cap[\mathcal{U}]}_{\alpha} \subseteq \mathbf{\Pi}^{\mathcal{U}}_{(\rho+1)\vee\alpha} \quad A \cap \left[\mathbf{\Delta}^{\mathcal{U}}_{\alpha}\right] \subseteq \mathbf{\Delta}^{A\cap[\mathcal{U}]}_{\alpha} \subseteq \mathbf{\Delta}^{\mathcal{U}}_{(\rho+1)\vee\alpha}$$ ◿

Let $\mathcal{U} \subseteq \wp X$, $\mathcal{V} \subseteq \wp Y$. When $\mathcal{V}$ is countable, define:

$$\mathrm{rank}^{\mathcal{U},\mathcal{V}} : \mathbf{Meas}(\langle X, \sigma(\mathcal{U})\rangle, \langle Y, \sigma(\mathcal{V})\rangle) \to \omega_1$$
$$\mathrm{rank} f := \bigvee_{A \in \mathcal{V}} f^{-1}[A]$$

Let $f : \langle X, \sigma(\mathcal{U})\rangle \to \langle Y, \sigma(\mathcal{V})\rangle$ be a measurable function.

◸**A.8.** What's the rank of a continuous function between two topological spaces? ◿

◸**A.9.** Bound the rank of $f^{-1}[A]$ for every $A \in \sigma(\mathcal{V})$, using $\mathrm{rank} A$ and $\mathrm{rank} f$.
Is your bound tight enough to deduce that $\mathrm{rank} f^{-1}[A] \le \mathrm{rank} A$ when f is continuous for the topologies generated by $\mathcal{U}$ and $\mathcal{V}$? ◿

Let $\mathcal{U} \subseteq \wp(C \times X)$ and $\mathcal{V} \subseteq \wp X$ be two classes of subsets. We will regard subsets $[\![-]\!] \in \mathcal{U}$ as potential encodings for subsets in $\mathcal{V}$, where each element $c \in C$ encodes the section subset $[\![c]\!] := \{x \in X | x \in [\![c]\!]\}$.
We say that $[\![-]\!] \in \mathcal{U}$ is a *$\mathcal{U}$-$\mathcal{V}$-encoder* when $\mathcal{V} = \{[\![c]\!] | c \in C\}$. The intended meaning is that such an encoder lets us cover all the $\mathcal{V}$-subsets with a code in C. The literature uses the term *C-universal set for* $\mathbf{\Xi}$ for a $\mathcal{U}$-$\mathcal{V}$-encoder, when $\mathcal{U}$ and $\mathcal{V}$ belong to the same family of subset classes Ξ, such as $\mathcal{U} = \mathbf{\Sigma}_{\alpha}(C \times X)$ and $\mathcal{V} = \mathbf{\Sigma}_{\alpha}(X)$.

◸**A.10.** Show that if $[\![-]\!]$ is a $\mathcal{U}$-$\mathcal{V}$-encoder, then $[\![-]\!]^{\mathsf{C}}$ is a $[\mathcal{U}]^{\mathsf{C}}$-$[\mathcal{U}]^{\mathsf{C}}$-encoder. ◿

◸**A.11.** Let $[\![-]\!]$ be a $\mathcal{U}$-$\mathcal{V}$ encoder, where $\mathcal{U} \subseteq \wp(C \times C)$ and $\mathcal{V} \subseteq \wp C$. Consider the diagonal function $\vartriangle := \lambda x. \langle x, x\rangle : C \to C \times C$.
Show that $\vartriangle^{-1}[[\![-]\!]^{\mathsf{C}}] \notin \mathcal{V}$. ◿

We'll use this diagonalisation technique to show that the Borel hierarchy doesn't collapse for the reals.

◸**A.12.** Recall the Cantor space $\mathbb{C} \subseteq \mathbb{R}$, let $\mathcal{V}$ be the open subsets of $\mathbb{R}$, let $\mathcal{V}' := \mathbb{C} \cap [\mathcal{U}]$ be the open subsets in $\mathbb{C}$, and $\mathcal{U}'$ be the open subsets of $\mathbb{C} \times \mathbb{C}$.

- Show that if, for all $1 \le \alpha < \omega_1$, we have $\mathbf{\Sigma}^{\mathcal{V}'}_{\alpha} \ne \mathbf{\Pi}^{\mathcal{V}'}_{\alpha}$, then $\mathbf{\Sigma}^{\mathcal{V}}_{\alpha} \ne \mathbf{\Pi}^{\mathcal{V}}_{\alpha}$ too, and so the Borel hierarchy for $\mathbb{R}$ only stabilises at ω_1.
- Show that if $\mathbb{G}$ has a $\mathbf{\Sigma}^{\mathcal{U}'}_{\alpha}$-$\mathbf{\Sigma}^{\mathcal{V}'}_{\alpha}$-encoder, then $\mathbf{\Sigma}^{\mathcal{V}'}_{\alpha} \ne \mathbf{\Pi}^{\mathcal{V}'}_{\alpha}$.
- Show that, for all $1 \le \alpha \in \omega_1$, $\mathbb{G}$ has both a $\mathbf{\Sigma}^{\mathcal{U}'}_{\alpha}$-$\mathbf{\Sigma}^{\mathcal{V}'}_{\alpha}$ encoder and a $\mathbf{\Pi}^{\mathcal{U}'}_{\alpha}$-$\mathbf{\Pi}^{\mathcal{V}'}_{\alpha}$ encoder. ◿

The last exercise constructs a non-Borel set. This result doesn't fit the narrative, but we've already introduced most of the tools required for the job.

◸**A.13.** A Borel set is *analytic* when it is empty, or a continuous image of the Baire space $\mathbb{Y} := \mathbb{N}^{\mathbb{N}}$. We denote by $\mathbf{\Sigma}^1_1(S)$ the class of analytic subsets of S. One can show that every Borel set is analytic, but that would require a lot of additional machinery.

- Show that if $\mathcal{B}_{\mathbb{Y}} \subseteq \mathbf{\Sigma}^1_1(\mathbb{Y})$ and we have a $\mathbf{\Sigma}^1_1(\mathbb{Y}\times\mathbb{Y})$-$\mathbf{\Sigma}^1_1(\mathbb{Y})$-encoder, then $\mathcal{B}_{\mathbb{Y}} \subset \mathbf{\Sigma}^1_1(\mathbb{Y})$.
- Show that we have a $\mathbf{\Pi}^0_1(\mathbb{Y}\times\mathbb{Y})$-$\mathbf{\Pi}^0_1(\mathbb{Y})$-encoder.
- Construct a homeomorphism $\mathbb{Y} \cong \mathbb{Y}\times\mathbb{Y}$. Derive a $\mathbf{\Pi}^0_1(\mathbb{Y}\times\mathbb{Y}\times\mathbb{Y})$-$\mathbf{\Pi}^0_1(\mathbb{Y}\times\mathbb{Y})$-encoder $\mathcal{F}[\![-]\!]$.
- Show that setting $x \in [\![c]\!]$ when $\exists z. \langle x, z\rangle \in \mathcal{F}[\![c]\!]$ is an $\mathbf{\Sigma}^1_1(\mathbb{Y}\times\mathbb{Y})$-$\mathbf{\Sigma}^1_1(\mathbb{Y})$-encoder.
  Hint: the graph of a continuous function over $\mathbb{Y}$ is a $\mathbf{\Pi}^0_1(\mathbb{Y}\times\mathbb{Y})$ set. ◿

B Lebesgue measurability

Measure theory is based on measurable sets, and the Borel sets of real numbers is the minimal collection of these sets. While the Borel sets are closed under many operations, they are not closed under all of them, and measure theorists and descriptive set theorists investigate other, more general, classes of subsets: analytic sets, universally measurable sets, and the Lebesgue sets. Nonetheless, in this batch of exercises we'll see that the extra level of generality Lebesgue measurability offers, which subsumes the other notions, doesn't get around Aumann's theorem: classical measure theory seems incompatible with function-spaces.

In the process, we'll use measures, measure spaces, and the Lebesgue measurable sets. These concepts come up in the context of higher-order measure theory, and these exercises may serve as classical tutorial to these concepts.

An *outer measure* $\boldsymbol{\lambda}^*$ on a set X is a function $\boldsymbol{\lambda}^* : \wp X \to \mathbb{W}$, i.e., an assignment of a non-negative, potentially infinite, real value to *every* subset, that is moreover *monotonically σ-subadditive*: for every countable set of subsets $I \subseteq_{\aleph_1} \wp X$, and every $A \subseteq \bigcup_{B \in I} B$, we have $\boldsymbol{\lambda}^* A \leq \sum_{B \in I} \boldsymbol{\lambda}^* B$.

◸**B.1.** Let $\boldsymbol{\lambda}^*$ be an outer measure on a set X. Show:

- The empty set has null outer measure: $\boldsymbol{\lambda}^* \varnothing = 0$.
- Monotonicity: $A \subseteq B \implies \boldsymbol{\lambda}^* A \leq \boldsymbol{\lambda}^* B$.
- σ-subadditivity: for every countably infinite family of subset $\vec{A} \in (\wp X)^{\mathbb{N}}$ we have $\boldsymbol{\lambda}^* \left(\bigcup_{i \in \mathbb{N}} A_i\right) \leq \sum_{i \in \mathbb{N}} \boldsymbol{\lambda}^* A_i$.
- Every function $\boldsymbol{\lambda}^* : \wp X \to \mathbb{W}$ satisfying these three conditions is an outer measure. ◿

A *measure* $\boldsymbol{\lambda}$ on a measurable space X is a non-negative, σ-additive function, i.e., for every countable set I and I-indexed family of pairwise-disjoint measurable sets $\langle U_i \in \mathcal{B}_X \rangle_{i \in I}$, we have: $\boldsymbol{\lambda}\left(\bigcup_{i \in I} U_i\right) = \sum_{i \in I} \boldsymbol{\lambda} U_i$. A *measure space* $\Omega = \left\langle \llcorner \underset{\mathbf{Meas}}{\Omega} \lrcorner, \boldsymbol{\lambda}_\Omega \right\rangle$ is a measurable space $\llcorner \underset{\mathbf{Meas}}{\Omega} \lrcorner$ and a measure on it, and similarly an *outer measure* space is a measurable space with an outer measure on it.

Every measure space has an outer measure space on its sets of points. This is the only example of interest. Let Ω be a measure space. Define a function $\boldsymbol{\lambda}^*_\Omega : \wp \llcorner \underset{\mathbf{Set}}{\Omega} \lrcorner \to \mathbb{W}$ by setting, for every $A \in \wp \llcorner \Omega \lrcorner$:

$$\boldsymbol{\lambda}^* A \coloneqq \inf \left\{ \boldsymbol{\lambda} U \middle| U \in \mathcal{B}_{\llcorner \underset{\mathbf{Meas}}{\Omega} \lrcorner}, U \supseteq A \right\}$$

So $\boldsymbol{\lambda}^* A$ is the least measure we can assign to A by approximating it from the outside with a measurable set. Hence the name — outer measure.

◸**B.2.** Show that $\boldsymbol{\lambda}^*_\Omega$ is an outer measure on $\llcorner \underset{\mathbf{Set}}{\Omega} \lrcorner$, and that it extends $\boldsymbol{\lambda}$: for every $U \in \mathcal{B}_\Omega$, we have $\boldsymbol{\lambda}^* U = \boldsymbol{\lambda} U$. ◿

◸**B.3.** Show that, for every $A \in \wp \llcorner \Omega \lrcorner$ there is some measurable subset $U \in \mathcal{B}_\Omega$, $U \supseteq A$, satisfying $\boldsymbol{\lambda}^* A = \boldsymbol{\lambda} U$. ◿

Let Ω be an outer measure space. A subset $E \subseteq \llcorner \Omega \lrcorner$ is *outer measurable* when, for every $A \subseteq \llcorner \Omega \lrcorner$ we have:

$$\boldsymbol{\lambda}^* A = \boldsymbol{\lambda}^* (A \cap E) + \boldsymbol{\lambda}^* (A \cap E^{\complement})$$

⊿**B.4.** Let Ω be an outer measure space. For every subset $E \subseteq \llcorner\Omega\lrcorner$, E is outer measurable iff for every $A \subseteq \llcorner\Omega\lrcorner$ we have: $\boldsymbol{\lambda}^* A \geq \boldsymbol{\lambda}^*(A \cap E) + \boldsymbol{\lambda}^*(A \cap E^{\mathsf{C}})$. ⊿

⊿**B.5.** Let Ω be a measure space. Show that every measurable set $U \in \mathcal{B}_\Omega$ is outer measurable in the associated outer measure space. ⊿

⊿**B.6.** Let Ω be an outer measure space.

- The outer measurable subsets of an outer measure space form a σ-algebra $\mathcal{G}_\Omega$.
- The outer measure $\boldsymbol{\lambda}^*$ restricts to a measure on $\left\langle \llcorner\Omega\lrcorner, \mathcal{G}_\Omega \right\rangle$.

We denote the resulting measure space by $\overline{\overline{\Omega}} \coloneqq \left\langle \left\langle \llcorner\Omega\lrcorner, \mathcal{G}_\Omega \right\rangle, \boldsymbol{\lambda}^* \right\rangle$. ⊿

The *Lebesgue subsets* of $\mathbb{R}$ are the outer measurable subsets w.r.t. the Lebesgue measure. The process: measure space $\Omega \mapsto$ outer measure space $\left\langle \llcorner\Omega\lrcorner, \boldsymbol{\lambda}^* \right\rangle \mapsto$ measure space $\overline{\overline{\Omega}}$ seems like it enhances the space with many more measurable sets. What we'll show next is that these sets aren't too far off from the measurable sets we started with.
A *null* set in a measure space Ω is a subset $Z \subseteq \llcorner\Omega\lrcorner$ that is contained in a 0-measure set: there is some $U \in \mathcal{B}_\Omega$ with $Z \subseteq U$ and $\boldsymbol{\lambda} U = 0$. Let $\mathcal{N}_\Omega$ denote the set of $\boldsymbol{\lambda}$-null sets.

⊿**B.7.** The null subsets form an ideal: If Z is a null set and $U \subseteq Z$ is any subset, then U is also a null set. Therefore they are closed under non-empty intersections. The null subsets are closed under countable unions. ⊿

⊿**B.8.** Consider the Borel space $\mathbb{R}$ and the Lebesgue measure $\boldsymbol{\lambda}$. Show that there is a $\boldsymbol{\lambda}$-null set that is not Borel measurable. ⊿

⊿**B.9.** Show that every null set is outer measurable. ⊿

⊿**B.10.** Let Ω be a measure space. Prove Ω and $\overline{\overline{\Omega}}$ have the same null sets: $\mathcal{N}_\Omega = \mathcal{N}_{\overline{\overline{\Omega}}}$. ⊿
Let Ω be a measure space. A *negligible* measurable subset is a measurable subset $U \in \mathcal{B}_\Omega$ such that, for every measurable subset $V \subseteq U$, we have $\boldsymbol{\lambda} V = 0$ or $\boldsymbol{\lambda} V = \infty$. Non-null negligible measurable subsets are sometimes called 'atomic sets of infinite measure', and Vákár and Ong (2018) call the negligible sets 0-∞-sets. While it may seem strange to call a set of potentially infinite measure negligible, in the context of *integration*, a Lebesgue integrable function must vanish almost everywhere on negligible sets:

⊿**B.11.** Let U be a negligible measurable subset in a measure space Ω. Let $\varphi : \Omega \to \mathbb{W}$ be a Lebesgue integrable random variable, i.e., a function with a finite expectation $\int \boldsymbol{\lambda}\varphi < \infty$. Show that $\boldsymbol{\lambda}\{\omega \in U | \varphi\omega \neq 0\} = 0$. ⊿
A *negligible* subset is a set contained in a negligible measurable subset, and we denote the set of negligible subsets by $\overline{\mathcal{N}}_\Omega$.

⊿**B.12.** Let Ω be a measure space and consider the scaled measure $\infty \odot \boldsymbol{\lambda}$. Show that:

- Every measurable set U is negligible in the scaled measure, and therefore every subset is negligible.
- A subset is null in the scaled measure iff it is null in Ω. ⊿

⊿**B.13.** Consider the Lebesgue measure on $\mathbb{R}$. Show every negligible subset is null. ⊿

⊿**B.14.** The negligible subsets generalise the null sets and have analogous properties:

- The negligible subsets form an ideal.
- The negligible subsets are closed under countable unions.
- Every negligible subset is outer measurable.
- Every negligible subset of finite outer measure is null. ⊿

The *completion* of a measure space Ω is the following measurable space $\overline{\Omega}$:

- It has the same points $\llcorner \overline{\langle X, \boldsymbol{\lambda} \rangle} \lrcorner := \llcorner X \lrcorner$.
- Its σ-algebra is generated by the measurable sets and the null sets: $\mathcal{B}_{\overline{\Omega}} := \sigma(\mathcal{B}_X \cup \mathcal{N})$.

◸**B.15.** Show that the following are equivalent for a subset $U \subseteq \llcorner \Omega \lrcorner$:

- U is measurable in the completion $\overline{\Omega}$
- There is a measurable set $V \in \mathcal{B}_\Omega$ and a null set $Z \in \mathcal{N}_\Omega$ such that $U = V \cup Z$.
- There is a measurable $V \in \mathcal{B}_X$ such that $U \setminus V$ is null. ⊿

◸**B.16.** Let E be an outer measurable subset in a measure space Ω. Show that if E has finite outer measure, then:

- There are measurable $U, V \in \mathcal{B}_\Omega$ with $U \subseteq E \subseteq V$ and $\boldsymbol{\lambda} U = \boldsymbol{\lambda}^* E = \boldsymbol{\lambda} V$.
- $E = E_{\mathcal{B}} \cup E_{\mathcal{N}}$ where $E_{\mathcal{B}} \in \mathcal{B}_\Omega$ and $E_{\mathcal{N}} \in \mathcal{N}$. ⊿

A measure space Ω is *σ-finite* when there is a countable measurable partition $\llcorner \Omega \lrcorner = \biguplus_{i \in I} \Omega_i$ for which every subset has $\boldsymbol{\lambda} \Omega_i = 0$.

◸**B.17.** Show that in a σ-finite space Ω, the outer measurable sets coincide with the completion σ-algebra: $\mathcal{B}_{\overline{\overline{\Omega}}} = \mathcal{B}_{\overline{\Omega}}$ ⊿

Let $\mathbb{R}_{\boldsymbol{\lambda}}$ be the measurable space over the reals with the Lebesgue σ-algebra. By the last few exercises, every Lebesgue measurable set on the reals is a Borel set apart from a null set of points. Similarly, a Lebesgue measurable function $f : \llcorner \mathbb{R}_{\boldsymbol{\lambda}} \lrcorner_{\mathbf{Meas}} \to \mathbb{R}$ is almost-everywhere equal to a Borel measurable function $g : \mathbb{R} \to \mathbb{R}$:

◸**B.18.** Let X be a measurable space whose σ-algebra is *countably generated*, i.e., there is a countable set $\mathcal{U} \subseteq \mathcal{B}_X$ such that $\mathcal{B}_X = \sigma(\mathcal{U})$. For every Lebesgue measurable function $f : \mathbb{R}_{\boldsymbol{\lambda}} \to X$ there is a Borel measurable function $g : \mathbb{R} \to X$ such that $f(x) = g(x)$ $\boldsymbol{\lambda}(\mathrm{d}x)$-almost certainly. ⊿

So the class of Lebesgue measurable functions is not profoundly different from the class of Borel measurable functions, especially as far as integration is concerned.
We are now ready to prove the Lebesgue-measurable version of Aumann's theorem:

▸ **Theorem** (Aumann's theorem for Lebesgue measurable evaluation)**.** *There is no σ-algebra on $\mathcal{B}_{\mathbb{R}}$ making the membership relation $[- \in -] : \mathcal{B}_{\mathbb{R}} \times \mathbb{R}_{\boldsymbol{\lambda}} \to 2$ measurable. Similarly, there is no σ-algebra on $\mathbf{Meas}(\mathbb{R}, \mathbb{R})$ making evaluation $\mathrm{eval} : \mathbf{Meas}(\mathbb{R}, \mathbb{R}) \times \mathbb{R}_{\boldsymbol{\lambda}} \to \mathbb{R}$ measurable.*

It suffices prove that the discrete σ-algebra on $\mathcal{B}_{\mathbb{R}}$ doesn't make the membership predicate measurable:

◸**B.19.** Assume that $[- \in -] : \mathcal{B}_{\mathbb{R}} \times \mathbb{R}_{\boldsymbol{\lambda}} \to 2$ is not measurable when we equip $\mathcal{B}_{\mathbb{R}}$ with the discrete σ-algebra. Show the following.

- The membership predicate is not measurable w.r.t. every σ-algebra on $\mathcal{B}_{\mathbb{R}}$.

- Evaluation eval : $\mathbf{Meas}(\mathbb{R},\mathbb{R}) \times \mathbb{R}_{\boldsymbol{\lambda}} \to \mathbb{R}$ is not measurable w.r.t. every σ-algebra on $\mathbf{Meas}(\mathbb{R},\mathbb{R})$. ◿

From this point, we assume to the contrary that $[- \in -] : \mathcal{B}_{\mathbb{R}} \times \mathbb{R}_{\boldsymbol{\lambda}} \to 2$ is measurable. Let:

$$\mathcal{U}_0 := [- \in -]^{-1}[\mathbf{true}] = \{\langle U, x\rangle \in \mathcal{B}_{\mathbb{R}} \times \mathbb{R} | x \in U\} \in \mathcal{B}_{\mathcal{B}_{\mathbb{R}} \times \mathbb{R}_{\boldsymbol{\lambda}}} = \wp\mathcal{B}_{\mathbb{R}} \otimes (\mathcal{B}_{\mathbb{R}} \cup \mathcal{N}_{\mathbb{R}})$$

Let $e_{\mathcal{B}} : \boldsymbol{b} \twoheadrightarrow \mathcal{B}_{\mathbb{R}}, e_{\mathcal{N}} : \boldsymbol{n} \twoheadrightarrow \mathcal{N}_{\mathbb{R}}$, and $e_{\wp} : \boldsymbol{p} \twoheadrightarrow \wp\mathcal{B}_{\mathbb{R}}$ be enumerations of the Borel sets, null sets, and powerset-over-Borel-sets of reals, respectively. Then we also have an enumeration of a generating family for the box σ-algebra of the product space $\mathcal{B}_{\mathbb{R}} \times \mathbb{R}_{\boldsymbol{\lambda}}$:

$$e : \boldsymbol{p} \times (\boldsymbol{b} \uplus \boldsymbol{n}) \twoheadrightarrow [\wp\mathcal{B}_{\mathbb{R}}] \times [\mathcal{B}_{\mathbb{R}} \cup \mathcal{N}_{\mathbb{R}}] := \{\mathcal{U} \times E | \mathcal{U} \subseteq \mathcal{B}_{\mathbb{R}}, E \in \mathcal{B}_{\mathbb{R}} \cup \mathcal{N}_{\mathbb{R}}\}$$
$$e := (e_{\wp}(\pi_1 -)) \times ([e_{\mathcal{B}}, e_{\mathcal{N}}]\,(\pi_2 -))$$

By Ex.A.4 the σ-term interpretation function $[\![-]\!]\,e$ is surjective, and so there is some σ-term t such that $\mathcal{U}_0 = [\![t]\!]\,e$. Let $V_0 := \operatorname{supp} t$, and then $V_0 \subseteq \boldsymbol{p} \times (\boldsymbol{b} \uplus \boldsymbol{n})$ is a countable enumeration of the variable names that appear in t, and we may restrict e to $e_0 : V_0 \to [\wp\mathcal{B}_{\mathbb{R}}] \times [\mathcal{B}_{\mathbb{R}} \cup \mathcal{N}_{\mathbb{R}}]$, and consider t as a term t_0 over V_0 such that $[\![t_0]\!]\,e_0 = \mathcal{U}_0$.
Let:

- $N := \bigcup_{(p,\iota_2 n) \in V_0} e_{\mathcal{N}} \in \mathcal{N}$;
- Z a 0-measure Borel set with $N \subseteq Z$; and
- $S := \mathbb{R} \setminus Z$ equipped with the Borel-subspace σ-algebra.

◸**B.20.** Show that N is indeed a null set, so that Z exists. Show that S is an uncountable Borel set. ◿

Define $e_1 : V_0 \to \wp\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_S$ by setting:

$$e_1(p, \iota_1 b) := (e_{\wp} p) \times (e_{\mathcal{B}} b \cap S) \qquad e_1(p, \iota_2 n) := \emptyset$$

By re-interpreting the σ-term t_0 with e_1, we have a measurable set $[\![t_0]\!]\,e_1 \in \mathcal{B}_{\mathcal{B}_{\mathbb{R}} \times S}$.

◸**B.21.** Show that for every $U \in \mathcal{B}_{\mathbb{R}}$ and $s \in S$, we have $\langle U, s\rangle \in [\![t_0]\!]\,e_0$ iff $\langle U, s\rangle \in [\![t_0]\!]\,e_1$. ◿

The last ingredient is to note that, by the original Aumann's theorem, there is no σ-algebra on $\mathcal{B}_S$ that makes the membership predicate $[- \in -] : \mathcal{B}_S \times \mathbb{R} \to 2$ measurable.

◸**B.22.** Use this last fact to get the desired contradiction. ◿

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