A convenient fibration for dependently-typed probability theory

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Abstract

We describe semantic structures relevant for interpreting dependent types for statistical and probabilistic modelling. Our development extends the theory of quasi-Borel spaces (qbses) of Staton et. al, which support simply-typed, higher-order probability theory with continuous distributions. It is well-known that qbses can interpret a dependent-type theory supporting dependent function-spaces through the codomain fibration. We define an equivalent split fibration based on the family fibration, which we call quasi-Borel families (qbfs), characterise its structure, equip it with fibred monads of measures and probability, and use them to develop dependently-typed probability theory.

We characterise the structure of the qbf fibration that is relevant for dependently-typed probability theory in elementary form. Our characterisations include: context extension, dependent pairs, dependent functions, extensional identity types, fibred products and coproducts, subspaces, a universe of propositions, and straightforward internalisation and externalisation principles for discrete spaces. We use these concepts to define fibred distribution and probability monads, the semantic structure needed to interpret probability distributions under a dependent context. We show that this structure satisfies a fibred version of Kock's synthetic measure theory. We also use these concepts to develop a qbs counterpart to Kolmogorov's conditional expectation. Our main result is a version of the conditional expectation that, under standard regularity assumptions, is measurable in both the random variables we are conditioning, and the observation map we are conditioning by.

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1 Introduction

Modern probability theory and statistics require both discrete and continuous probability distributions, e.g., a pair of fair dice and a uniformly distributed real number in the interval $[0, 1]$. The classical foundation for these concepts is measure theory. In this work, we contribute another stepping stone towards a type-rich semantic foundation for statistics and probability, emphasising dependent types. Our work adds to some recent interest in semantic structure for dependent types in probabilistic modelling [33, 34, 27, 32], and here we offer a unique combination of advantages. First, our semantic structures are **strict**. Semantic structures for dependently-typed concepts simplify significantly when they are stable under semantic substitution, in a sense we will make precise later. Second, we support unrestricted dependent-function spaces. Finally, like St Clere Smithe [33], our semantic structures form a conservative extension of standard Borel probability theory—standard Borel spaces and measurable functions are a full subcategory of our semantic universe of discourse. Thus, we can access the full range of probability theoretic results developed to date.

Our development extends the recently proposed theory of **quasi Borel spaces** [14], or **qbs**es for short. Unlike measurable spaces, qbses support well-behaved function spaces for all qbses, allowing us to study concepts such as distributions over random variables or Borel subsets directly. By construction, the category of qbses is a (Grothendieck) quasitopos, implying it is locally Cartesian closed, and so supports a dependently-typed semantics through the codomain fibration. Though by now much of the simply-typed structure of qbses has been explored [14, 29, 9, 39, 28], its dependently-typed structure has been unexplored.

Contribution. We divide this work into two parts. In Part I, we present a convenient fibration that supports the standard suite of structure expected of dependently-typed probability: context extension and semantics for extensional propositional equality, dependent pairs and dependent functions, fibred coproducts and subspaces, and fibred monads for (s-finite) distributions and probability distributions. In Part II, we show that structures inherent and relevant to modern probability theory organise into these dependently-typed spaces.

By 'convenient fibration', we mean three different properties. The traditional technical meaning of 'convenience' [38], is that it supports **well-behaved constructions** (products, function spaces, etc.), which in the dependently-typed setting amount to dependent-pair spaces, dependent-function spaces, and various fibred versions of the convenient constructions, and indeed the fibration we describe supports those. That is not surprising: as a quasitopos, the codomain fibration $\mathrm{cod} : \mathbf{Qbs}^{\rightarrow} \to \mathbf{Qbs}$ has the appropriate semantic structure for dependent types. In early parts of this work, we have developed the theorems in Part II using the codomain fibration. We found the codomain fibration inconvenient for two reasons:

Strictness. The codomain fibration is not split. As Hofmann [15] argues, non-strictness is inconvenient. We describe an equivalent split fibration concretely and directly.

{In/ex}ternalisation. We show that the semantic structure of this fibration supports a smooth transition between two modes of reasoning: internal and external.

In addition to this fibration, we present two fibred monads for statistics and probability. One fibred monad for (s-finite) distributions, which supports Bayesian conditioning [36]. Here we simplified the development conceptually. This simplicity is misleading: it follows a lengthy trial-and-error process, and form a substantial contribution. We show that these two monads validate the axioms of Kock's so-called synthetic measure theory [22], lending us abstract assurance that they support standard, but fibred, probabilistic reasoning.

In Part II, we develop a susbstantial use-case for this rich semantic structure by investigating spaces of integrable random variables, the so-called Lebesgue spaces. These spaces, which cannot be given a measurable space structure in the classical theory due to Aumann's theorem [16], form the foundation of much modern probability and statistics, including stochastic processes, inferential statistics, causal modelling, and so on. We focus the development on the conditional expectation, which underlies all these topics.

Our main result is that under standard regularity assumptions, there is a measurable version of the conditional expectation:

$$\Gamma \vdash \mathbb{E}_{-}[-|-] : (\mu : \mathbf{P}\Omega) \to (H : \Omega \to \Theta) \to (f : \mathcal{L}^1(\Omega, \mu)) \to \{g : \mathcal{L}^1(\Theta, \mu_H) | g = \mathbb{E}_\mu[f|H]\}$$

Here, $\mathbb{E}_{-}[-|-]$ is our notation for this measurable conditional expectation function. It is a dependent function, and it depends on the **observation map** H from the sample space Ω to an **observation space** Θ. Our main result: this dependent function is measurable in all aguments, including the higher-order H, going beyond classical measure theory.

We proceed as follows. Sec. 2 briefly reviews relevant preliminaries. Sec. 3 starts Part I by presenting the convenient fibration and its fibred equivalence to the codomain fibration. Sec. 4 characterises the semantic structure needed for dependently-typed contexts/judgements, and Sec. 5 characterises the semantic structure needed for dependent types. Both sections are quite detailed in order to facilitate follow-up work, but may be read more quickly or skimmed at first reading to reach the more exciting later sections. Sec. 6 presents the fibred distribution monad and its fibred probability submonad. Sec. 7 starts Part II, arranging standard probabilistic concepts such as random variables and their operations into dependently-typed spaces. Sec. 8 establishes many technical concepts and results we use in order to prove our main theorem. It can be skipped on first reading. Sec. 9 is the final technical section, establishing our main theorem. Secs. 10–11 end, with brief comparison with related work and concluding remarks. Appendix A includes a short introduction to Grothendieck fibrations and the fibred equivalence between the codomain and family fibration over **Set**.

2 Preliminaries

We recall standard background, fixing notation and terminology.

2.1 Quasi-Borel spaces

We will move between various notions of **space** X, all of which will comprise of a pair $\langle \underline{X}, \mathcal{S}_X \rangle$ consisting of: a set $\underline{X}$ and some structure $\mathcal{S}_X$ over it. We call $\underline{X}$ the **carrier** set, and its elements the **points** in the space.

Measure theory. Recall that a **field** $\mathcal{B}$ of subsets over $\underline{X}$ is a Boolean sub-algebra $\mathcal{B} \subseteq \mathcal{P}(\underline{X})$ of the powerset, i.e., it contains the empty set $\emptyset$, the total set $\underline{X}$, and complements $E^{\complement}$, unions $E \cup F$, and intersections $E \cap F$ of subsets $E, F \in \mathcal{B}$. A **σ-field** $\mathcal{B}$ is such a field that is moreover closed under countable unions $\bigcup_{i \in I} E_i$, whenever $I \subseteq \mathbb{N}$ is a countable set and $E_{-} : I \to \mathcal{B}$ is an I-indexed sequence of $\mathcal{B}$-elements. Measure theoretic accounts sometimes use the terms **algebra** and **σ-algebra**, but we will not. Elements E of (σ-)fields are called **events**. The **Borel subsets** $\mathcal{B}_{\mathbb{R}}$ of the real line $\underline{\mathbb{R}} \coloneqq (-\infty, \infty)$ consist of the smallest σ-field containing all the open intervals (a, b). Thus, for example, it contains the closed intervals $[a, b]$, and the half-open intervals $[a, b)$ and $(a, b]$.

constants:
$$\frac{x \in \underline{X}}{\underline{x} \coloneqq (\lambda r.x) \in \mathcal{R}}$$

precomposition:
$$\frac{\alpha \in \mathcal{R} \qquad \varphi \in \mathbf{Meas}(\mathbb{R}, \mathbb{R})}{(\alpha \circ \varphi) \in \mathcal{R}}$$

recombination:
$$\frac{I \subseteq \mathbb{N} \qquad E_{-} : I \to \mathcal{B}_{\mathbb{R}} \qquad \alpha_{-} : I \to \mathcal{R} \qquad \underline{\mathbb{R}} = \biguplus_{i \in I} E_i}{[E_i.\alpha_i]_{i \in I} \coloneqq (\lambda r \in E_i.\alpha_i r) \in \mathcal{R}}$$

Figure 1 metaphorology axioms

A **measurable space** M is a pair $\langle \underline{M}, \mathcal{B}_M \rangle$ consisting of a set $\underline{M}$ equipped with a σ-field $\mathcal{B}_{\underline{M}} \subseteq \mathcal{P}(\underline{M})$ over it. Thus the reals equipped with their Borel subsets form a measurable space $\mathbb{R}$. A **measurable function** $f : M \to K$ between measurable spaces M and K is a function between their corresponding sets of points $f : \underline{M} \to \underline{K}$ whose inverse function sends events in K to event in M, i.e.: $f^{-1}[-] : \mathcal{B}_M \leftarrow \mathcal{B}_K$. Every constant function is measurable, and since $\mathcal{B}_{\mathbb{R}}$ contains the open intervals, every continuous function is measurable $f : \mathbb{R} \to \mathbb{R}$. Every identity $\mathrm{id} : M \to M$ is measurable, and measurable functions are closed under composition, and we denote the category of measurable spaces and measurable functions by **Meas**. We will not require any further measure theoretic results, and either derive them directly using the theory of quasi-Borel spaces, or refer to standard accounts for a measure-theoretic proof that transfers without change to quasi-Borel spaces.

The category **Meas** supports many relevant constructions, such as products and coproducts, subspaces, and quotients. Crucially, it does not support function spaces [16], let alone dependent-function spaces. We will not use it further.

Quasi-Borel spaces. A **metaphorology**[1] $\mathcal{R}$ over a set $\underline{X}$ consists of a subset of the functions from the real number line to the set $\mathcal{R} \subseteq (\underline{\mathbb{R}} \to \underline{X})$, which we call **(admissible) random elements** (res), satisfying the three axioms in Fig. 1.

A metaphorology axiomatises which functions can be admitted as random elements of a probability space over the real line. The **constants** axiom states that every point admits a constant random element, representing a point mass at that element. The **precomposition** axiom states that we can admit the random elements that measurably transforms the real line before any admissible random element. The **recombination** axiom states that we can admit as a random element the pasting together of countably many random elements, according to a countable partition of the real line into Borel subsets. The metaphorology of the real line consists of the Borel-measurable functions as random elements: $\mathcal{R}_{\mathbb{R}} \coloneqq \mathbf{Meas}(\mathbb{R}, \mathbb{R})$. The smallest metaphorology over a set $\underline{X}$ consists of the $\boldsymbol{\sigma}$**-simple** functions—recombinations of constant functions, and the largest metaphorology consists of all functions.

A **quasi-Borel space** (qbs) X is a pair $\langle \underline{X}, \mathcal{R}_X \rangle$ consisting of a set of points and a metaphorology over it. Thus the real line $\underline{\mathbb{R}}$ equipped with its metaphorology $\mathcal{R}_{\mathbb{R}}$ form a qbs $\mathbb{R}$. A **(quasi-)measurable function** $f : X \to Y$ between qbses is a function between their corresponding sets of points $f : \underline{X} \to \underline{Y}$ that sends res to res by postcomposition: $(f\circ) \coloneqq (\lambda\alpha.f \circ \alpha) : \mathcal{R}_X \to \mathcal{R}_Y$. Every constant function is quasi-measurable, and the measurable functions $f : \mathbb{R} \to \mathbb{R}$ coincide exactly with the quasi-measurable functions

[1] Going back to its original roots, 'metaphor' originates from the Greek $\mu\varepsilon\tau\alpha$ ('meta', across) and $\varphi\varepsilon\rho\omega$ ('phero', to carry). This choice makes 'metaphors' an appealing alternative terminology to 'random element', which we will not adopt.

$$\textbf{fibred constants:}\quad \frac{\gamma \in \underline{\Gamma} \qquad a \in \underline{A}_\gamma}{\underline{a} \in \mathcal{R}^{\underline{\gamma}}} \qquad \textbf{fibred precomposition:}\quad \frac{\alpha \in \mathcal{R}^{\boldsymbol{v}} \qquad \varphi \in \mathbf{Qbs}(\mathbb{R}, \mathbb{R})}{(\alpha \circ \varphi) \in \mathcal{R}^{\boldsymbol{v} \circ \varphi}}$$

$$\textbf{fibred recombination:}\quad \frac{\begin{array}{cc} I \subseteq \mathbb{N} & E_- : I \to \mathcal{B}_{\mathbb{R}} \\ \mathbb{R} = \biguplus_{i \in I} E_i & \alpha_- : (i : I) \to \mathcal{R}^{\boldsymbol{v}_i} \end{array}}{[E_i.\alpha_i]_{i \in I} \in \mathcal{R}^{[E_i.\boldsymbol{v}_i]_{i \in I}}}$$

Figure 2 Axioms for fibred metaphorologies

$f : \mathbb{R} \to \mathbb{R}$. From this point we will deal almost exclusively with qbses and quasi-measurable functions, and so omit the qualification 'quasi-'. Identity functions are measurable and composition of measurable functions between qbses is measurable, and we denote the category of quasi-Borel spaces and measurable functions by **Qbs**.

The category **Qbs** originated as a Grothendieck quasitopos, as a category of separated sheaves on standard Borel spaces [14, 19, Thm. A.4.4.5]. While we do not directly rely on this fact in our presentation, it underlies and permeates our development thoroughly, and we will remark in passing about its relevance. Most importantly, **Qbs** has products, coproducts, a strong notion of subspaces and quotients, and it is locally Cartesian closed. To keep this introductory section short, we will review each of these concepts as we characterise its dependently-typed counterpart.

We conclude the preliminaries here, covering more topics as needed.

Part I

Semantic foundations

Just as the family fibration splits the codomain fibration for **Set**, we define in Sec. 3 a split fibration that splits the codomain fibration for **Qbs**. By generalities, the codomain fibration has the required structure [19, Thm. A.4.4.5], and our contribution in Secs. 4–5 is to characterise it concretely. Since we have much to cover, we keep the development brief. We expect most readers to skip Secs. 4–5, and only refer to them as needed. This part concludes with the fibred monads in Sec. 6.

3 Quasi-Borel families

We define a split fibration over **Qbs** that is fibred equivalent to the codomain fibration over **Qbs**. We reuse the family fibration over **Set**, which allows us to defer much of the complexity of dependent types to the meta-language, and focus on measurability concerns.

▶ **Definition 1.** *Let Γ be a qbs and $\underline{\Gamma} \vdash \underline{A}$ a family indexed by its points. A Γ-**fibred metaphorology** $\mathcal{R}_\Gamma \vdash \mathcal{R}$ over $\underline{A}$ is a family of subsets of dependent functions $\mathcal{R}^{\boldsymbol{v}}_{\underline{A}} \subseteq (r : \mathbb{R}) \to \underline{A}_{\boldsymbol{v}\, r}$, indexed by res $\boldsymbol{v} \in \mathcal{R}_\Gamma$, whose elements we call **(admissible) fibred random elements** (fres), satisfying the three axioms in Fig. 2.*

We use the dependent-function notation $(r : \mathbb{R}) \to \underline{A}_{\boldsymbol{v}\, r}$ rather than the more set-theoretic $\prod_{r \in \mathbb{R}} \underline{A}_{\boldsymbol{v}\, r}$ purely as a matter of taste.

In a way we will make precise later (Thm. 13), a metaphorology is exactly the structure we need to make the Grothendieck construction $\coprod_{\underline{\Gamma}} \underline{A}$—the set of dependent pairs $\langle \gamma, a \rangle$

with $\gamma \in \underline{\Gamma}$ and $a \in \underline{A}_\gamma$—into a qbs $\coprod_\Gamma A$, so that the dependency/display map $\langle \gamma, a\rangle \mapsto \gamma : \coprod_\Gamma A \to \Gamma$ measurable. Fibred random elements $\alpha \in \mathcal{R}^{\boldsymbol{v}}$ over a random element $\boldsymbol{v} \in \mathcal{R}_\Gamma$ correspond to all the random elements in $\beta \in \mathcal{R}_{\coprod_\Gamma A}$, via $\beta r = \langle \boldsymbol{v} r, \alpha r\rangle$.

The three axioms for fibred metaphorologies explicate how to admit fibred random elements over the random element admitted by the corresponding axiom. **Fibred constants** admits constant functions fibred over the constant re. **Fibred precomposition** admits the fre that first measurably transforms the real line before applying a fre, fibred over the correspondingly transformed re. **Fibred recombination** admits a countable recombination of fres fibred over the corresponding recombined re.

▶ **Definition 2.** *A **quasi-Borel family** (qbf) A **over** Γ is a pair $\left\langle A, \mathcal{R}^{-}_{A}\right\rangle$ consisting of a family $\underline{\Gamma} \vdash A$ indexed by the points of Γ equipped with a Γ-fibred metaphorology $\mathcal{R}_A$ over $\underline{A}$. A **quasi-Borel family** is a pair $\langle \Gamma, A\rangle$ consisting of a qbs Γ and a qbf over it.*

As with families, we write $\Gamma \vdash A$ for the qbf $\langle \Gamma, A\rangle$.

▶ **Example 3.** Every qbs X becomes a qbf $\Gamma \vdash \underline{X}$ by ignoring the dependency, i.e., by taking the family $\underline{X}_\gamma \coloneqq X$ and the fibred metaphorology: $\mathcal{R}^{\boldsymbol{v}}_{\underline{X}} \coloneqq \mathcal{R}_X$.

▶ **Example 4.** Let $\overset{A}{\underset{\Gamma}{d\downarrow}}$ be a qbs morphism. Define its **preimage** qbf $\Gamma \vdash d^{-1}[-]$ as the family $d^{-1}[\gamma] \coloneqq \{a \in \underline{A} | d\, a = \gamma\}$ with the fibred metaphorology $\mathcal{R}^{\boldsymbol{v}}_{\Gamma\vdash d^{-1}[-]} \coloneqq (d\circ)^{-1}[\boldsymbol{v}]$ over it.

Some readers may benefit from the following two perspectives on qbfs—suggested to us by the anonymous referees. The first perspective uses a universe of sets $\mathcal{U}$. A qbf is a pair of qbses $\langle \Gamma, A\rangle$ where the carrier of A is a subset $\underline{A} \subseteq \Gamma \times \mathcal{U}$, and the first projection $\pi_1 : A \to \Gamma$ is measurable. This perspective emphasises the graph of the family as well as its display map, pre-selecting which pullbacks to use to interpret substitution. The second perspective further uses a universe of propositions **Prop**, and explicates indexing. A qbf (Γ, A) is then a function $\underline{A} : \underline{\Gamma} \to \mathcal{U}$—a $\mathcal{U}$-set indexed by the carrier $\underline{\Gamma}$—together with a function $\mathcal{R}_{\Gamma\vdash A} : (\alpha : \mathcal{R}_\Gamma) \to ((r : \mathbb{R}) \to A(\alpha, r)) \to$ **Prop**. This reformulation makes the dependency structure explicit, aligning with some presentations of type theory. We prefer not to make universes explicit in the definition, leaving the precise mathematical interpretation of the family assignment $\gamma \mapsto \underline{A}_\gamma$ to the reader's foundation of choice.

▶ **Definition 5.** *A **qbf map** $(\theta \vdash f) : (\Gamma \vdash A) \to (\Delta \vdash B)$ between two qbfs is a pair $\langle \theta, f\rangle$ consisting of a map between the underlying families $(\theta \vdash f) : (\underline{\Gamma} \vdash \underline{A}) \to (\underline{\Delta} \vdash \underline{B})$ such that: $\forall \boldsymbol{v} \in \mathcal{R}_\Gamma. \theta \circ \boldsymbol{v} \in \mathcal{R}_\Delta$ and $\forall \alpha \in \mathcal{R}^{\boldsymbol{v}}_{\Gamma\vdash A}. f \overset{\boldsymbol{v}}{\circ} \alpha \in \mathcal{R}^{\theta\circ\boldsymbol{v}}_{\Delta\vdash B}$.*

The first condition states that $\theta : \Gamma \to \Delta$ is a qbs morphism.

▶ **Example 6.** Measurable functions $f : X \to Y$ yield qbf maps: $\Gamma \vdash \underline{f} : \underline{X} \to \underline{Y}$ via $\underline{f}_\gamma \coloneqq f$.

To make notation more succinct, we will write $\gamma : \Gamma \vdash A_\gamma$ for the qbf $\Gamma \vdash \langle A_\gamma\rangle_{\gamma\in\Gamma}$. Similarly, we write $\lambda\gamma.\theta(\gamma) \vdash \lambda a.f_\gamma a$ for the qbf map $(\lambda\gamma.\theta(\gamma)) \vdash \langle \lambda a.f_\gamma a\rangle_\gamma$, i.e., the leftmost occurrence of γ is binding for the whole term, not just θ avoiding repetition.

▶ **Example 7.** Consider a commuting square in **Qbs**, i.e., a morphism $\langle \theta, f\rangle : \overset{A}{\underset{\Gamma}{d\downarrow}} \to \overset{B}{\underset{\Delta}{e\downarrow}}$ in the arrow category $\mathbf{Qbs}^{\to}$. Applying the preimage functor to the underlying commuting square of functions gives a qbf map:

$$\left(\lambda\gamma.\theta(\gamma) \vdash f|_{d^{-1}[\gamma]})\right) : \left(\Gamma \vdash d^{-1}[-]\right) \to \left(\Delta \vdash e^{-1}[-]\right)$$

Although one can show it preserves fres directly, this fact follows from the upcoming theorem about the fibred equivalence to the codomain fibration.

The identity qbf map is the identity for the underlying family. Qbf maps compose by as their underlying family maps. We then get the category **Qbf** of qbfs and their maps.

▶ **Example 8.** We have the following functors: $\underline{-}$: **Qbs** → (Γ ⊢) sending each qbs to the dependency ignoring qbf; Index : **Qbf** → **Qbs** sending each qbf to its indexing qbs: $(\Gamma \vdash A) \mapsto \Gamma$; and qbf map to its qbs morphism $\langle\theta, f\rangle \mapsto \theta$. We lift the Grothendieck construction to qbfs by defining $\underline{\coprod_\Gamma A} \coloneqq \coprod_{\underline{\Gamma}} \underline{A}$ and $\mathcal{R}_{\coprod_\Gamma A} \coloneqq \{\langle \boldsymbol{v}, \alpha\rangle | \boldsymbol{v} \in \mathcal{R}_\Gamma, \alpha \in \mathcal{R}_A^{\boldsymbol{v}}\}$ and their maps: $\coprod_\theta f \coloneqq \lambda \langle\gamma, a\rangle . \langle\theta\gamma, f_\gamma a\rangle$. We also have a forgetful functor $\underline{-}$: **Qbf** → **Fam** sending a qbf over an indexing qbs to its underlying family over the carrier: $(\Gamma \vdash A) \mapsto (\underline{\Gamma} \vdash \underline{A})$, and a qbf map to the underlying family map.

The previous example is the central semantic structure of our work:

▶ **Definition 9.** *The* ***quasi-Borel family fibration*** *is the functor* Index : **Qbf** → **Qbs***.*

We will also often project one of the fibre and treat it as a qbs:

▶ **Example 10.** Let Γ be a qbs. Each index $\gamma \in \Gamma$ induces a functor $-_\gamma : (\Gamma \vdash) \to$ **Qbs** sending each qbf A and vertical map $f : A \to B$ over Γ to their **fibre at** γ:

$$\underline{A_\gamma} \coloneqq \underline{A}_\gamma \qquad \mathcal{R}_{A_\gamma} \coloneqq \mathcal{R}_A^\gamma \qquad f_\gamma : A_\gamma \to B_\gamma$$

We first characterise the Cartesian qbf maps:

▶ **Proposition 11.** *For* $(\theta \vdash f) : (\Gamma \vdash A) \to (\Delta \vdash B)$ *in* **Qbf***, TFAE: (1)* $\theta \vdash f$ *is Cartesian; (2) the square* $\mathrm{dep}_{\theta\vdash f} \in \mathbf{Qbf}^{\to}$ *is a pullback square in* **Qbs***; and (3) the following two conditions hold: (a) for all* $\gamma \in \underline{\Gamma}$*, the* γ*-fibre* $f_\gamma : A_\gamma \to B_{\theta\gamma}$ *is bijective; and (b) for all* $\boldsymbol{v} \in \mathcal{R}_\Gamma$ *and* $\alpha : (r : \mathbb{R}) \to A_{\boldsymbol{v}\,r}$*, we have:*$\alpha \in \mathcal{R}_A^\gamma \iff (f \overset{\boldsymbol{v}}{\circ} \alpha) \in \mathcal{R}_B^{\theta\circ\boldsymbol{v}}$*.*

Next, we choose the cleavage that will split the qbf fibration:

▶ **Proposition 12.** *For every qbs* Γ*, qbf* $\Delta \vdash B$*, and qbs morphism* $\theta : \Gamma \to \Delta$*:*

- *We have a qbf* $\Gamma \vdash B[\theta]$ *given by:* $\underline{B[\theta]}_\gamma \coloneqq \underline{B}_{\theta\,\gamma}$ *and* $\mathcal{R}_{B[\theta]}^{\boldsymbol{v}} \coloneqq \mathcal{R}_B^{\theta\circ\boldsymbol{v}}$*.*
- *The qbf map* $\left(\lambda\gamma.\theta\gamma \vdash \mathrm{strengthen}_\gamma^{\theta,B} \coloneqq \lambda b.b\right) : (\Gamma \vdash B[\theta]) \to (\Delta \vdash B)$ *is Cartesian.*

We therefore have a Grothendieck fibration equipped with the cleavage $-[\theta]$. The following theorem packs together all the goals we set out to achieve in this section:

▶ **Theorem 13.** *The qbf fibration equipped with its cleavage is split. The forgetful functors* $\underline{-}$: **Qbf** → **Fam** *and* $\underline{-}$: **Qbs** → **Set** *form a split functor from the qbf fibration to the family fibration. The dependency functor* dep : $\mathbf{Qbf} \to \mathbf{Qbs}^{\to}$ *induced by the Grothendieck construction and the preimage functor form a fibred equivalence between the qbf and codomain fibrations. Together with the forgetful functor* $\mathbf{Qbs}^{\to} \to \mathbf{Set}^{\to}$*, the fibred equivalence over* **Qbs** *lifts the fibred equivalence over* **Set***.*

Since we know that **Qbf** has preliminary structure to interpret type dependency:

▶ **Corollary 14.** *The split qbf fibration has a* ***full comprehension category*** *structure [17, Def. 10.4.2]. In detail, consider the* ***comprehension*** *functor* dep : $\mathbf{Qbf} \to \mathbf{Qbs}^{\to}$*. Then:*

- *the comprehension functor lifts the family fibration along the codomain fibration; and*
- *this functor is full and faithful, and preserves Cartesian morphisms.*

This comprehension functor is a fibred equivalence, and since the codomain fibration possesses much of the required dependently-typed structure, we can transport it to the qbf fibration. With a little more abstract work, through Hofmann's [15] results, we can show that the semantic structure is preserved on the nose. However, as we'll see in Part II, we will benefit from characterising this semantic structure in elementary terms. We will therefore spend the next couple of section spelling it out and mentioning its properties.

4 Context structure

In Part II, we will be using dependently-typed judgements under hypotheses such as:

$$\frac{\Gamma \vdash M : [1,\infty) \qquad \Gamma \vdash K : \Omega \to \Theta \qquad \Gamma, \mu : \mathbf{P}\Omega \vdash V : \mathcal{L}^M(\Omega,\mu)}{\Gamma, \mu : \mathbf{P}\Omega \vdash \sqrt[M]{\mathbb{E}_\mu \left[|V| \,\middle|\, K \right]} : \mathcal{L}^M(\Theta, \mu_K)}$$

Our goal is to characterise the structure needed to interpret this notation semantically. As we will see, the semantic structure will be close enough to its meta-theoretic interpretation that setting up an formal calculus and its interpretation will seem as significant pedantry that is out of scope for this work. As usual in dependently typed settings, we divide this task into two parts. In this section, we deal with the structure of dependent judgements, including: fibred terminal objects, context extension through comprehension, and the semantics of propositional equality. In the next section, characterise common dependently-typed constructions for qbfs. The technical development is involved and dry. Readers may skip or skim to Sec. 6.

4.1 Terms as local sections

Fix $\mathbb{1} := \{\langle\rangle\}$ as a chosen terminal set, which uniquely determines a terminal qbs: the carrier is the same singleton, determining the metaphorology as $\mathcal{R}_{\mathbb{1}} := \left\{\underline{\langle\rangle}\right\}$. For each qbs Γ, define the **fibred terminal qbf** by $\Gamma \vdash \mathbb{1} := \underline{\mathbb{1}}$. Vertical qbf maps $\Gamma \vdash M : \mathbb{1} \to A$ then amount to a sequence picking out an element $M_\gamma\langle\rangle \in A_\gamma$ in each fibre. We will therefore call such maps **terms**, and smoothly transition between the notation $\Gamma \vdash M : A$, the local section $\langle M_\gamma \langle\rangle\rangle_{\gamma \in \underline{\Gamma}}$, and the qbf map $\Gamma \vdash M : \mathbb{1} \to A$. The following result expresses the fact that this qbf satisfies the correct properties to interpret terms:

▶ **Proposition 15.** *The fibred terminal qbf together with the identity* $\mathrm{id} : \mathsf{Index}(- \vdash \mathbb{1}) = \mathrm{Id}_{\mathbf{Qbs}}$ *form a split right adjoint to the qbf fibration as a split functor to itself from the identity fibration.*

Take a measurable function $\theta : \Gamma \to \Delta$ and a term $\Delta \vdash M : B$. The reindexing functor sends M to the term $\Gamma \vdash M[\theta] : B_{\theta -}$ corresponding to the section $\langle M_{\theta\gamma}\langle\rangle\rangle_{\gamma\in\Gamma}$.

▶ **Example 16.** Terms reindex the fibres of a qbf $\Delta \vdash A$ as qbfs in another context. Since Δ is a qbs, we can treat it as a qbf in any another context $\Gamma \vdash \Delta$. The **internal fibre** qbf is:

$$\frac{\Delta \vdash A \qquad \Gamma \vdash M : \Delta}{\Gamma \vdash A_M} \qquad \underline{A_M}_\gamma := \underline{A}_{M_\gamma\langle\rangle} \qquad \mathcal{R}^{\boldsymbol{v}}_{A_M} := \mathcal{R}_A^{\lambda r. M_{\boldsymbol{v}\, r}\langle\rangle}$$

4.2 Context extension

Given a qbf $\Gamma \vdash A$. The **context extension of** Γ **by** $x : A$ is the qbs $(\Gamma, x : A) := (\coprod_\Gamma A)$ given by the Grothendieck construction. To emphasise we are interpreting the Grothendieck construction as context extension, we will denote its elements $\langle \gamma, a\rangle$ by $\gamma, x \mapsto a$. The associated **weakening** is the measurable function:

$$\mathsf{weaken}^{\Gamma\vdash x:A} : (\Gamma, x : A) \xrightarrow{\text{dep}} \Gamma \qquad \mathsf{weaken}\,\langle\gamma, a\rangle := \gamma$$

Extended contexts have a term $\Gamma, x : A \vdash x := \left\langle a \in \underline{A}_\gamma \right\rangle_{\gamma, x\mapsto a} : A_{\mathsf{weaken}}$ for the new 'variable' x. Post-composing the term x with the Cartesian morphism over weakening, $\mathsf{weaken}^{\Gamma\vdash x:A} \vdash$

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash K : A \qquad \Gamma, x : A \vdash B}{\Gamma \vdash (M =: x \coloneqq K) \times B} \qquad \frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : B_{-,x\mapsto M}}{\Gamma \vdash \mathsf{refl}\, M, N : (M =: x \coloneqq M) \times B}$$

$$\frac{\Gamma, x : A \vdash B \quad \Gamma, y : A \vdash C \quad \Gamma \vdash M, K : A \quad \Gamma \vdash f : B_{-,x\mapsto M} \to C_{-,y\mapsto M}}{\Gamma \vdash \mathbf{match}\, - \,\mathbf{with}\, \{f\} : (M =: x \coloneqq K) \times B \to C_{-,y\mapsto K}}$$

$$\underline{(M =: x \coloneqq K) \times B}_{\gamma} \coloneqq \begin{cases} M_\gamma\langle\rangle =: a \coloneqq K_\gamma : & B_{\gamma,x\mapsto a} \\ \text{otherwise:} & \emptyset \end{cases} \qquad \mathcal{R}^{\boldsymbol{v}}_{(M=:x:=K)\times B} \coloneqq \left\{ \begin{matrix}\beta : (r : \mathbb{R}) \to \\ B_{\boldsymbol{v}r,x\mapsto\alpha\, r}\end{matrix} \middle| \begin{matrix}\exists\alpha \in \mathcal{R}^{\boldsymbol{v}}_A.\beta \in \mathcal{R}^{\boldsymbol{v},x\mapsto\alpha}_{\Gamma,x:A\vdash B} \\ \forall r \in \mathbb{R}. M_{\boldsymbol{v}r}\langle\rangle = \alpha\, r = K_{\boldsymbol{v}r}\langle\rangle\end{matrix} \right\}$$

$$(\mathbf{match}\, - \,\mathbf{with}\, \{f\})_\gamma \coloneqq \begin{cases} M_\gamma\langle\rangle = K_\gamma\langle\rangle : & f_\gamma : B_{\gamma,x\mapsto M_\gamma} \to C_{\gamma,y\mapsto M_\gamma} \\ \text{otherwise :} & [] : \emptyset \to C_{\gamma,y\mapsto K_\gamma\langle\rangle} \end{cases}$$

$$(\mathsf{refl}\, M, N)_\gamma\langle\rangle \coloneqq N_{\gamma,x\mapsto M_\gamma\langle\rangle} \in B_{\gamma,x\mapsto M_\gamma\langle\rangle} = \underline{(M =: x \coloneqq M) \times B}_{\gamma}$$

Figure 3 qbfs and maps for propositional fording

strengthen, gives the qbf map:

$$\left(\mathsf{weaken} \vdash \mathsf{val}_{\Gamma\vdash x:A} \coloneqq \langle\lambda\langle\rangle.a\rangle_{\gamma,x\mapsto a}\right) : (\Gamma, x : A \vdash \mathbb{1}) \to (\Gamma \vdash A)$$

While we rarely make use of the map val directly, we need it to characterise the correct semantic structure for context extension:

▶ **Proposition 17.** *The assignments* $(\Gamma \vdash A) \mapsto (\Gamma, x : -)$ *together with the counit* val *form a right adjoint to the fibred terminal object* $(- \vdash \mathbb{1}) : \mathbf{Qbs} \to \mathbf{Qbf}$*. Chaining this adjunction with the universal property of Cartesian morphisms, the mate of a term* $\Gamma \vdash M : A_{\theta-}$ *is the measurable function* $(\lambda\gamma.\theta\gamma, x \mapsto M_\gamma\langle\rangle) : \Gamma \to (\Gamma, x : A)$*.*

4.3 Semantics for propositional equality

The final contextual structure we need is the semantics of **propositional equality**. Although we will mostly gloss over the issues that require the use of propositional equality, we expect future work investigating syntactic development to use this semantic structure. This structure lets us denote a qbf for reasoning about equality between terms that type-checkers for those calculi will not be able to discharge. The standard categorical structure needed to model identity types is typically a certain left adjoint to reindexing along the diagonal. **Propositional fording**[2], depicted in Fig. 3, describes this left adjoint. The forded B is a qbf that may depend on the joint value we substitute for x, and the fibres of $(M =: x \coloneqq K) \times B$ are: B-elements in fibres for which M and K coincide; and empty otherwise. The fres over an indexing re $\boldsymbol{v} \in \mathcal{R}_\Gamma$ are those rfe in B of the form $\beta : (r : \mathbb{R}) \to B_{\boldsymbol{v}r,x\mapsto\alpha\, r}$ for some necessarily unique fre $\alpha : (r : \mathbb{R}) \to A_{\boldsymbol{v}r}$ taking the joint value $M_{\boldsymbol{v}r}\langle\rangle = \alpha\, r = K_{\boldsymbol{v}r}\langle\rangle$. Fording supports a reflexivity term refl: in a given fibre $\gamma \in \underline{\Gamma}$, we can pack elements in $B_{\gamma,x\mapsto M_\gamma}$ to

[2] McBride [26] coined the term 'fording' alluding to Henry Ford [11]. Fording allows the packaged B-elements to seemingly depend on an arbitrary x, so long as x is exactly the simultaneously identical elements denoted by M and K in the current fibre.

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash K : A}{\Gamma \vdash M =\!::\!= K} \qquad \underline{M =\!::\!= K}_{\gamma} := \begin{cases} M_{\gamma}\langle\rangle =: a := K_{\gamma} : & \mathbb{1} \\ \text{otherwise:} & \emptyset \end{cases}$$

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathsf{refl}\, M : M =\!::\!= M} \qquad \mathcal{R}^{\boldsymbol{v}}_{M=::=K} := \begin{cases} \forall r \in \mathbb{R}. M_{\boldsymbol{v}\,r}\langle\rangle = K_{\boldsymbol{v}\,r} : & \left\{\underline{\langle\rangle}\right\} \\ \text{otherwise:} & \emptyset \end{cases}$$

$$(\mathsf{refl}\, M)_{\gamma}\langle\rangle := \langle\rangle \in \mathbb{1}_{\gamma, x \mapsto M_{\gamma}\langle\rangle} = \underline{(M =: x := M) \times B}_{\gamma}$$

Figure 4 qbfs and maps for extensional propositional equality and propositional fording

get an element of $(M =: x := M) \times B$. The need for **match** $-$ **with** $\{-\}$ is more technical. The semantic structure for propositional equality requires the fording construct:

$$(y =: x := z) \times B : (\Gamma, y : A, z : A \vdash) \to (\Gamma, x : A \vdash)$$

to be a left adjoint to the following diagonal functor [17, Def. 9.3.5 and Exercise 10.3.3(iii-iv)]:

$$-[\mathrm{id}, y \mapsto x, z \mapsto x] : (\Gamma, y : A, z : A \vdash) \leftarrow (\Gamma, x : A \vdash)$$

The mate relation for this adjunction is given by **match** $-$ **with** $\{-\}$: it sends a vertical qbf map N from B to C defined in those fibres in which x and y are both M to a map out of B-elements packed under the proposition that $M =\!::\!= K$, and **transports** them to those C fibres with K for y.

We recover the propositional equality of Fig. 4 by fording the fibred terminal qbf as $(M =\!::\!= K) := (M =: x := K) \times \mathbb{1}$. Specialising the definitions for fording then yields the qbf which has trivial fibres, either terminals when the two terms coincide; and empty otherwise. The metaphorology includes fres only over res that identify M and K.

▶ **Proposition 18.** *Propositional fording $(y =: x := z) \times B$ equipped with the reflexivity maps $\mathsf{refl}\, -, x$ as a unit form a left adjoint to the diagonal functor. Moreover, they are strictly strong, and strictly satisfy the appropriate Beck-Chevalley condition.*

One repetitive aspect we will gloss over in the sequel is **local definitions**:

$$\frac{\Gamma \vdash M : A \qquad \Gamma, x : A, e : M =\!::\!= x \vdash K : B}{\Gamma \vdash \quad (\mathbf{let}\,(x : A) := M \;\mathbf{in}\; K) \quad := \quad K\,[\mathrm{id}, x \mapsto M, e \mapsto \mathsf{refl}\, M] \quad : B_{-, x \mapsto M, e \mapsto \mathsf{refl}\, M}}$$

They use propositional equality, where the body K may depend on x and its equality to M.

5 Constructing families

Now that we established the semantic structure needed for contexts and terms, we briefly characterise the relevant dependently-typed constructs that we use in the sequel. In this section we will focus on general-purpose constructions, and in the next Sec. 6 we construct fibred monads for measures and probability.

5.1 Dependent pairs

Dependent pairs—Σ-types—bundle dependent elements together.

Fig. 5 presents this construct and associated qbf maps. The family underlying the dependent pairing of the qbfs $\Gamma \vdash A$ and $\Gamma, x : A \vdash B$ is the dependent pairing of their

$$\frac{\Gamma \vdash A \qquad \Gamma, x : A \vdash B}{\Gamma \vdash (x : A) \times B}$$

$$\frac{\Gamma \vdash M : A \qquad \gamma : \Gamma \vdash K : B_{\gamma, x \mapsto M}}{\Gamma \vdash M, K : (x : A) \times B}$$

$$\frac{\Gamma \vdash M : (x : A) \times B \quad \Gamma, p : (x : A) \times B \vdash C \quad \gamma : \Gamma, a : A, b : B_{\gamma, x \mapsto a} \vdash K : C_{\gamma, p \mapsto a, b}}{\gamma : \Gamma \vdash \textbf{match}\, M\, \textbf{with}\ \{a, b \Rightarrow K\} : C_{\gamma, p \mapsto M}}$$

$$\frac{\Gamma, x : A \vdash B}{\Gamma, x : A \vdash (x_{,} -) : B \to ((x : A) \times B)\,[\mathsf{weaken}^{\Gamma \vdash x : A}]}$$

$$\frac{\Gamma \vdash C \qquad \Gamma, x : A \vdash f : B \to C\left[\mathsf{weaken}^{\Gamma \vdash x : A}\right]}{\Gamma \vdash \textbf{match} - \textbf{with}\ \{f\} : (x : A) \times B \to C}$$

$$\underline{(x : A) \times B}_{\gamma} \coloneqq (a : \underline{A}_{\gamma}) \times \underline{B}_{\gamma, x \mapsto a} \coloneqq \textstyle\coprod_{a \in A_{\gamma}} \underline{B}_{\gamma, x \mapsto a}$$

$$\mathcal{R}^{\boldsymbol{v}}_{(x:A)\times B} \coloneqq \{\lambda r.\alpha\, r, \beta\, r | \alpha \in \mathcal{R}^{\boldsymbol{v}}_{A}, \beta \in \mathcal{R}^{\boldsymbol{v}, x \mapsto \alpha}_{B}\}$$

$$(M, N)_{\gamma} \coloneqq \left\langle M_{\gamma}\langle\rangle, N_{\gamma, x \mapsto M_{\gamma}\langle\rangle}\langle\rangle\right\rangle \qquad (x_{,} -)_{\gamma, x \mapsto a} b \coloneqq \langle a, b\rangle$$

$$\left(\begin{matrix}\textbf{match} - \textbf{with}\\ \{a, b \Rightarrow K\}\end{matrix}\right)_{\gamma} \langle a', b'\rangle \coloneqq K_{\gamma, a \mapsto a', b \mapsto b'}\langle\rangle \qquad \textbf{match} - \textbf{with}\ \{f\})_{\gamma}\,\langle a, b\rangle \coloneqq f_{\gamma, x \mapsto a} b$$

Figure 5 qbf and maps for dependent pairs

underlying families, which itself is given fibrewise by the Grothendieck construction on the residual $\underline{A}_{\gamma}$-indexed family $\underline{B}_{\gamma, x \mapsto -}$. The fres over an indexing re $\boldsymbol{v}$ are correlated pointwise dependent pairings α, β of a fre α in A over $\boldsymbol{v}$ and a fre in B over the extended indexing re $\boldsymbol{v}, x \mapsto \alpha$. The rest of the structure is as in the set-theoretic family fibration. A technical point in our presentation is that the semantics constructions we use in the sequel which involves terms and a weak form of pattern matching, are slightly different than the standard semantic structure for dependent pairs, namely a left adjoint to reindexing along context extension with some additional properties [17, Prop. 10.3.3]. Our notation can still be reduced to the standard semantic structure, but inlines a few steps to a degree that it is better to be explicit about what we mean. The figure therefore includes both constructs, first the constructs $-, -$ and **match** $-$ **with** $\{a, b \Rightarrow -\}$ we will use in the sequel. Then, it presents the unit $x_{,} -$ for the left adjoint and the mate of the adjunction **match** $-$ **with** $\{-\}$.

▶ **Proposition 19.** *The constructions $(x : A) \times -$ equipped with the unit $(x_{,} -)$ are left adjoint to the reindexing functors: $-\left[\mathsf{weaken}^{\Gamma \vdash x : A}\right] : (\Gamma \vdash) \to (\Gamma, x : A \vdash)$. Moreover, they are strictly strong and strictly satisfy the appropriate Beck-Chevalley condition. The forgetful fibred functors to the family fibration over* **Set** *preserve this structure.*

5.2 Dependent functions

The qbfs for **dependent functions**—Π-types—set quasi-Borel spaces apart as a foundation for probability with continuous distribution that conservatively extends the standard theory. These qbfs will allow us to study spaces of integrable random variables in Part II.

Figure 6 presents this construct and associated qbf maps. The fibre at γ comprises of those dependent functions between the underlying families that send domain fre over the constant indexing re, to codomain fre over the indexing re extended by the given domain fre. The fre over a given indexing re is less intuitive, but two considerations motivate it. First,

$$\frac{\Gamma \vdash A \qquad \Gamma, x : A \vdash B}{\Gamma \vdash (x : A) \to B} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x : A.M : (x : A) \to B}$$

$$\frac{\Gamma \vdash M : (x : A) \to B \qquad \Gamma \vdash K : A}{\Gamma \vdash M\,K : B_{-,x\mapsto K}} \qquad (\lambda x : A.M)_\gamma\langle\rangle \coloneqq \lambda a.M_{\gamma,x\mapsto a}\langle\rangle$$

$$(M\,K)_\gamma \coloneqq (M_\gamma\langle\rangle)(K_\gamma\langle\rangle)$$

$$\frac{\Gamma \vdash A \qquad \Gamma, x : A \vdash B}{\Gamma, x : A \vdash \mathrm{eval}\ {-x} : ((x : A) \to B)\,[\mathsf{weaken}^{\Gamma\vdash x:A}] \to B} \qquad (\mathrm{eval}\ {-x})_{\gamma,x\mapsto a} f \coloneqq f\,a$$

$$\frac{\Gamma, x : A \vdash f : C\left[\mathsf{weaken}^{\Gamma\vdash x:A}\right] \to B}{\Gamma \vdash \mathrm{curry}\ f : C \to (x : A) \to B} \qquad (\mathrm{curry}\ f)_\gamma c \coloneqq \lambda a.f_{\gamma,x\mapsto a}$$

$$\underline{(x : A) \to B}_\gamma \coloneqq \left\{ f : (a : \underline{A}_\gamma) \to B_{\gamma,x\mapsto a} \,\middle|\, \forall \alpha \in \mathcal{R}^\gamma_A . f \circ \alpha \in \mathcal{R}^{\gamma,x\mapsto\alpha}_B \right\}$$

$$\mathcal{R}^{\boldsymbol{v}}_{(x:A)\to B} \coloneqq \left\{ \varphi : (r : \mathbb{R}) \to \underline{(x : A) \to B}_{\boldsymbol{v}\,r} \,\middle|\, \begin{array}{l} \forall \rho \in \mathbf{Qbs}(\mathbb{R}, \mathbb{R}), \alpha \in \mathcal{R}^{\boldsymbol{v}\circ\rho}_A . \\ \quad (\lambda r.((\varphi \circ \rho)\,r)(\alpha\,r)) \in \mathcal{R}^{\boldsymbol{v}\circ\rho,x\mapsto\alpha}_B \end{array} \right\}$$

Figure 6 qbf and maps for dependent functions

the res in $\mathbf{Qbs}$ exponential $X \to Y$ are given by currying the set of measurable functions $\mathcal{R}_{X\to Y} = \mathrm{curry}\,[\mathbf{Qbs}(\mathbb{R} \times X, Y)]$. Inlining all the conditions that guarantee a given function $\mathbb{R} \to \underline{X \to Y}$ gives the same condition as in Fig. 6, but without the indexing res. Another motivation comes from thinking about Kripke function spaces. The measurable function $\rho \in \mathbf{Qbs}(\mathbb{R}, \mathbb{R})$ acts as a witness to the accessibility relation. As with dependent pairs, we include both notational conventions that we use in the sequel, and the structure we need to characterise the semantic structure [17]. Both kinds of structure are standard, given by standard function abstraction, application, and currying.

▶ **Proposition 20.** *The constructions* $(x : A) \to -$ *equipped with the evaluation map as counit is right adjoint to the reindexing functors:* $-\left[\mathsf{weaken}^{\Gamma\vdash x:A}\right] : (\Gamma \vdash) \to (\Gamma, x : A \vdash)$. *Moreover, it strictly satisfies the appropriate Beck-Chevalley condition.*

The forgetful functor to $\mathbf{Fam}$ does not preserve this structure.

Proof. Establishing the recombination axiom is the one tricky part in the proof. Take any countable measurable partition of the reals $\mathbb{R} = \biguplus_{i\in I} E_i$ and I-indexed sequence of rfes $\varphi_i \in \mathcal{R}^{\boldsymbol{v}_i}_{(x:A)\to B}$. Recombine both the indexing res $\boldsymbol{v} \coloneqq [E_i.\boldsymbol{v}_i]_{i\in I}$ and the given fres $\varphi \coloneqq [E_i.\boldsymbol{v}_i]_{i\in I}$. Take any measurable $\rho \in \mathbf{Qbs}(\mathbb{R}, \mathbb{R})$ and rfe $\alpha \in \mathcal{R}^{\boldsymbol{v}\circ\rho}_A$. Define the events $F_i \coloneqq \rho^{-1}[E_i] \in \mathcal{B}_\mathbb{R}$, and indexing set $I_+ \coloneqq \{i \in I | F_i \neq \emptyset\}$. Let $\varepsilon : (i \in I_+) \to F_i$ be any countable choice function. For each $i \in I_+$, let $\sigma_i \coloneqq \left[F_i.\mathrm{id}, F^{\complement}_i.\underline{\varepsilon}_i\right] \in \mathbf{Qbs}(\mathbb{R}, \mathbb{R})$, $\alpha_i \coloneqq \alpha \circ \sigma_i$, and $\rho_i \coloneqq \rho \circ \sigma_i$, where σ_i recombines two res. Verify that $\boldsymbol{v} \circ \rho \circ \sigma_i = \boldsymbol{v}_i \circ \rho_i$ and:

$$(\lambda r.((\varphi \circ \rho)\,r)(\alpha\,r)) = [F_i.\lambda r.(\varphi_i \circ \rho_i\,r)(\alpha_i\,r)]_{i\in I_+} \qquad [F_i.\boldsymbol{v}_i \circ \rho_i, x \mapsto \alpha_i]_{i\in I_+} = \boldsymbol{v}\circ\rho, x \mapsto \alpha$$

These facts let us combine the premise of the recombination axiom with the recombination axiom for B, completing the proof. ◀

A simple, but powerful, use for dependent function qbfs is to internalise the fibred metaphorology as a qbf. This is analogous to the non-fibred case, where the qbs $\mathcal{R}_X \coloneqq \mathbb{R} \to X$ has X's metaphorology as carrier: $\underline{\mathcal{R}_X} = \mathcal{R}_X$.

▶ **Example 21.** Define the **internal fibred metaphorology** qbf:

$$\frac{\Gamma \vdash A}{\mathcal{R}_\Gamma \vdash \mathcal{R}^-_A \coloneqq (r : \mathbb{R}) \to A_{-r}} \qquad \forall \boldsymbol{v} \in \mathcal{R}_\Gamma.\quad \underline{\mathcal{R}^-_A}_{\boldsymbol{v}} = \mathcal{R}^{\boldsymbol{v}}_A$$

The containment $\supseteq$ follows from first principles. For the converse, take $\alpha \coloneqq \mathrm{id} \in \mathcal{R}^{\gamma}_{\mathbb{R}}$.

5.3 Ternalisation and fibred structure

As may have become clear by now, a dominant feature of the qbf fibration is that its semantic structure often lifts the structure from the family fibration, as with the contextual structure and dependent pairs. Even for dependent functions, that do not lift the underlying family of dependent functions, the structure is close, essentially restricting the underlying family by imposing a further measurability requirements. This property facilitates a smooth **externalisation** process. It allows us to study the set-theoretic structure of our constructed qbf, and reason directly about its inhabitants, for example to validate some equations. It also faciliates a smooth **internalisation** process. Often it is not onerous to discharge the cumulative measurability requirements needed to show that a given family map lifts to a qbf map. We call these two properties **ternalisation**.

Recall that a qbs is **discrete** when its metaphorology consists of only, hence all, **$\boldsymbol{\sigma}$-simple** functions, i.e., countable recombinations of constant functions. The discrete qbses over countable carriers are particularly important, since they are also **standard Borel spaces**, agreeing with classical (discrete) probability theory. When we index a qbf by a discrete qbs, ternalisation is frictionless:

▶ **Theorem 22** ({in/ex}ternalisation)**.** *Let I be a discrete qbs, and Γ a qbs. We have an isomorphism functor $\langle - [\mathrm{id}, i \mapsto j]\rangle_{j\in J} : (\Gamma, i : I \vdash) \xrightarrow{\cong} \prod_{j\in J}(\Gamma \vdash)$ from the fibre category to the product of fibres. We denote its inverse by $\Gamma, i : I \vdash -^{i\mapsto -}$.*

In particular, we have a bijection between: I-indexed families of qbfs $\left\langle \Gamma \vdash A^i \right\rangle_{i\in I}$; and qbfs $\Gamma, i : I \vdash A$, given by relating the carriers via $\underline{A}_{\gamma, i\mapsto j} = A^j_\gamma$ and the metaphorologies via $\mathcal{R}_A^{\boldsymbol{v}, i\mapsto \underline{j}} = \mathcal{R}_{A^j}\boldsymbol{v}$ and $\mathcal{R}_A^{\boldsymbol{v}, i\mapsto [E_j . \underline{i}_j]_{j\in J}} = \left\{ [E_j . \alpha_j]_{j\in J} \middle| \alpha : (j : J) \to \mathcal{R}^{\boldsymbol{v}}_{A^{i_j}} \right\}$.

*Internalisation is **externally strong**, i.e., we have an isomorphism:*

$$\mathrm{strength}^{-1}_{\Gamma,I} : (\Gamma, x : A^{i\mapsto -} \vdash) \xrightarrow[\langle \mathrm{id}, x[\mathrm{id}, i\mapsto j]\rangle_{j\in I}]{\cong} \prod_{i\in I} (\Gamma, x_i : A^i \vdash)$$

compatible with substitution—we have $\Gamma, i : I \vdash A^- [\theta]^{i\mapsto -} = A^{i\mapsto -}[\theta, i \mapsto i]$ for every $\theta : \Gamma \to \Delta$ and $\left\langle \Delta \vdash A^i \right\rangle_{i\in I}$.

One subtle consequence of compatibility with reindexing is that internalising a constant sequence of families is weakening, $(B)^{i\mapsto -} = B\left[\mathsf{weaken}^{i:I}\right]$:

$$(B\,[\mathsf{weaken}])\,[\mathrm{id}, i \mapsto j] = B\,[\mathsf{weaken} \circ (\mathrm{id}, i \mapsto j)] = B = (B)^{i\mapsto -}\,[\mathrm{id}, i \mapsto j]$$

We use this property when showing compatibility with context extension.

This theorem allows us, for example, to define qbfs and qbf maps indexed by the natural numbers by first defining their fibres or components externally, and internalising them. We

$$\frac{\Gamma \vdash A^- : I \to (\Gamma \vdash)}{\Gamma \vdash \coprod_{i\in I} A^i \coloneqq (i : I) \times A^{i\mapsto -}}$$

$$\underline{\coprod_{i\in I} A^i}_\gamma = \coprod_{i\in I} \underline{A^i}_\gamma$$

$$\frac{\Gamma \vdash M : A^j}{\Gamma \vdash \iota_j M \coloneqq j, M : (i : I) \times A^{i\mapsto -} \eqqcolon \coprod_{i\in I} A^i}$$

$$\mathcal{R}^{\boldsymbol{v}}_{\coprod_{i\in I} A^i} = \mathcal{R}^- \left\{\iota_i \circ \left[\mathcal{R}^{\boldsymbol{v}}_{A^i}\right] \middle| i \in I, \boldsymbol{v} \in \mathcal{R}_\Gamma\right\}$$

$$\Gamma \vdash \iota_j : A^j \xrightarrow{(i,-)[\mathrm{id}, i\mapsto j]} (i : I) \times A^{i\mapsto -} \eqqcolon \coprod_{i\in I} A^i$$

$$\frac{\Gamma \vdash M : A \qquad \text{for all } i \in I\text{: } \Gamma, x_i : A^i \vdash K_i : B}{\textbf{match}\, M\, \textbf{with}\ \{\iota_i x_i \Rightarrow K_i | i \in I\} \coloneqq \textbf{match}\, M\, \textbf{with}\ \left\{i, x \Rightarrow \text{strength}\, \langle K_i\rangle_{i\in I}\right\} : B}$$

$$\frac{\text{for all } i \in I\text{: } \Gamma \vdash f^i : A^i \to B}{\Gamma \vdash \left[f^i\right]_{i\in I} \coloneqq \textbf{match}\ -\ \textbf{with}\ \left\{f^{i\mapsto -}\right\} : \coprod_{i\in I} A^i \coloneqq (i : I) \times A^{i\mapsto -} \to B}$$

Figure 7 reducing the fibred coproduct structure to dependent pairs

will see many examples for this use in Part II. We will apply the Ternalisation Thm. 22 to construct fibred (co)products, starting with coproducts.

Figure 7 defines **fibred coproducts** i.e., coproducts in each fibre category $(\Gamma \vdash -)$. We define the coproduct in each fibre by internalising the sequence of qbfs and taking their dependent pair. We also include an explicit description of the underlying family and fibred metaphorology. The underlying family is the fibred coproduct of the underlying family of each component. The fibred metaphorology is the smallest one generated by embedding each component fibred metaphorology along the coproduct inclusion. We include both the operations on terms we use in the sequel, and the semantic structure needed, namely split fibred coproducts **strong** [17, Exercise 10.5.6]. For **match** $-$ **with** $\{-\}$, we use the external strength to internalise the branches into a dependent branch.

▶ **Proposition 23.** *The construction $\coprod_{i\in I} A^i$ equipped with the fibred coproduct injections forms strong split fibred coproducts. Fibred cotupling is the mate for the internalised qbf map. The forgetful fibred functors into the family fibration strictly preserves fibred coproducts.*

Turning to fibred products, Figure 8 proceeds similarly, using dependent functions from the discrete indexing qbf. These provide the fibred products [17, Def. 1.8.1].

▶ **Proposition 24.** *The construction $\bigtimes_{i\in I} A^i$ equipped with the fibred product projections forms split fibred products. The fibred tupling is given by currying the internalised qbf map. The forgetful fibred functor into the family fibration strictly preserves the fibred products.*

Finally, the fibres and metaphorology of the fibred exponentials are given by the dependent product without dependency:

$$\underline{A \to B}_\gamma = \mathbf{Qbs}(A_\gamma, B_\gamma) \qquad \mathcal{R}^{\boldsymbol{v}}_{A\to B} = \left\{\varphi \middle| \begin{array}{l}\forall \rho \in \mathbf{Qbs}(\mathbb{R}, \mathbb{R}).\forall \alpha \in \mathcal{R}^{\boldsymbol{v}\circ\rho}_A.\\ (\lambda r.((\varphi \circ \rho)r)(\alpha\, r)) \in \mathcal{R}^{\boldsymbol{v}\circ\rho}_B\end{array}\right\}$$

I.e., the function space fibres comprise the measurable functions from between the fibres.

$$\frac{\Gamma \vdash A^{-} : I \to (\Gamma \vdash)}{\Gamma \vdash \bigtimes_{i \in I} A^i \coloneqq (i : I) \to A^{i \mapsto -}} \qquad \underline{\bigtimes_{i \in I} A^i}_{\gamma} = \bigtimes_{i \in I} \underline{A^i}_{\gamma}$$

$$\mathcal{R}^{\boldsymbol{v}}_{\bigtimes_{i \in I} A^i} = \left\{ \langle \alpha_i \rangle_{i \in I} \middle| \forall i. \alpha_i \in \mathcal{R}^{\boldsymbol{v}}_{A^i} \right\}$$

$$\frac{\Gamma \vdash M : \bigtimes_{i \in I} A^i}{\Gamma \vdash \pi_j M \coloneqq M\, j : A^j} \qquad \Gamma \vdash \pi_j : \bigtimes_{i \in I} A^i \coloneqq (i : I) \to A^{i \mapsto -} \xrightarrow{(\mathrm{eval}\ -i)[\mathrm{id}, i \mapsto j]} A^j$$

$$\frac{\text{for all } i \in I\text{: } \Gamma, x_i : A^i \vdash M_i : A^i}{\langle M_i \rangle_{i \in I} \coloneqq \lambda i : I. M^{i \mapsto -} : (i : I) \to A^i \eqqcolon \bigtimes_{i \in I} A^i_i} \qquad \frac{\text{for all } i \in I\text{: } \Gamma \vdash f^i : B \to A^i}{\Gamma \vdash \langle f^i \rangle_{i \in I} \coloneqq \mathrm{curry}\, \langle f - \rangle^{i \mapsto -} : B \to (i : I) \to A^{i \mapsto -} \eqqcolon \bigtimes_{i \in I} A^i}$$

Figure 8 Reducing the fibred product structure to dependent functions

5.4 Subspaces

The final construction we consider lets us restrict attention to fibred subspaces of interest. For example, we will construct the qbf of Lebesgue integrable rvs we study in Part II as a fibred subspace of the space of all rvs.

Recall that a **subspace embedding** $e : X \hookrightarrow Y$ between qbses is a measurable function $e : X \to Y$ such that, for every re $\beta \in \mathcal{R}_Y$, if β lifts along e pointwise: $\forall r \in \mathbb{R}. \exists x \in \underline{X}. e\, x = \beta\, r$, then there exists a unique re $\alpha \in \mathcal{R}_X$ that lifts β along e, i.e.: $\beta = e \circ \alpha$. Subspace embeddings are injective—apply uniqueness to the constant res $\underline{x}_1, \underline{x}_2 \in \mathcal{R}_X$. Category theoretically, subspace embeddings are the strong monomorphisms in **Qbs**.

▶ **Example 25.** For every qbf $\Gamma \vdash A$ and index $\gamma \in \underline{\Gamma}$, the injection of the fibre into the the Grothendieck construction is a subspace embedding: $(\lambda a. \langle \gamma, a \rangle) : A_\gamma \hookrightarrow \coprod_{\Gamma} A$.

The qbs of **propositions** **Prop** has the Booleans as points and all functions as res, i.e. the largest metaphorology over its points. Its set-theoretic complete Boolean algebra structure is then measurable:

$$\underline{\mathbf{Prop}} \coloneqq \{\mathbf{true}, \mathbf{false}\} \qquad \neg : \mathbf{Prop} \to \mathbf{Prop}$$
$$\mathcal{R}_{\mathbf{Prop}} = (\mathbb{R} \to \underline{\mathbf{Prop}}) \qquad \bigwedge_{i:I}, \bigvee_{i:I} : (\bigtimes_{i \in I} \mathbf{Prop}) \to \mathbf{Prop}$$

A **predicate** φ over a qbf $\Gamma \vdash A$ is a proposition $\Gamma, x : A \vdash \varphi : \mathbf{Prop}$. We can treat propositions as 'types' with propositional equality, and since every set-theoretic function is measurable as a **Prop**, bounded quantification over predicates is measurable:

$$\frac{\Gamma \vdash \varphi : \mathbf{Prop}}{\Gamma \vdash \varphi \coloneqq (\varphi =\!::= \mathbf{true})} \qquad \frac{\Gamma, x : A \vdash \varphi : \mathbf{Prop}}{\Gamma \vdash (\forall x : A. \varphi), (\exists x : A. \varphi) : \mathbf{Prop}}$$

The usual proof system of constructive logic then become, through the standard propositions-as-types and proofs-as-programs interpretation, term constructors of the corresponding propositions. We also have the law of excluded middle available to us as a qbf map $\Gamma \vdash \mathsf{LEM} \coloneqq \langle\rangle : \varphi \vee \neg\varphi$ for every proposition $\Gamma \vdash \varphi : \mathbf{Prop}$.

▶ **Definition 26.** *A* ***fibred subspace embedding*** *between two qbfs* $\Gamma \vdash e : A \hookrightarrow B$ *is a vertical map* $\Gamma \vdash e : A \to B$ *satisfying, for every indexing re* $\boldsymbol{v} \in \mathcal{R}_\Gamma$ *and re* $\beta \in \mathcal{R}^{\boldsymbol{v}}_B$*, if* $\forall r : \mathbb{R}. \exists a : A. e\, a = \beta\, r$*, then there exists a unique* $\alpha \in \mathcal{R}_{\boldsymbol{v}}$ *that lifts* β *along* e*, i.e.:* $\beta = e \overset{\boldsymbol{v}}{\circ} \alpha$*.*

Ex. 25 generalises to fibred coproducts and dependent pairs:

▶ **Example 27.** The fibred coproduct injections are subspace embeddings $\Gamma \vdash \iota_j : X^j \hookrightarrow \coprod_{i\in I} X^i$. This fact generalises to dependent pairs in two ways. First, $\Gamma, x : A \vdash (x, -) : B \hookrightarrow (x : A) \times B$. Second, if $\Gamma, x : A \vdash M : B$, then we have the following term: $\Gamma \vdash (\lambda x : A.x, M) : A \to (x : A) \times B$ The resulting vertical function is a subspace embedding $\Gamma \vdash (\lambda a.a, M_a\langle\rangle) : A \hookrightarrow (x : A) \times B$.

▶ **Lemma 28** (Basic properties of fibred subspaces). *A vertical map $\Gamma \vdash e : A \to B$ is a fibred subspace embedding iff its Grothendieck construction $\coprod_\Gamma e : \coprod_\Gamma A \to \coprod_\Gamma B$ is a subspace embedding. Reindexing of fibred subspace embeddings are fibred subspace embedding. The qbs map $\Gamma \vdash$ **true** $: \mathbb{1} \to$ **Prop**, considered as a qbf, classifies fibred subspace embeddings via fibred pullback along predicates.*

We use the universal property for the fibred pullback resulting from the subspace classifier to define maps into subspaces (esp. for $C \coloneqq \mathbb{1}$):

$$\frac{\Gamma \vdash e : A \hookrightarrow B \qquad \Gamma \vdash f : C \to B \qquad \Gamma \vdash \mathit{prf} : (\forall z : C.\exists x : A.e\, x = f\, z)}{\Gamma \vdash \mathsf{lift}\ f\ \mathsf{along}\ e\ \mathsf{since}\ \mathit{prf} : C \to A}$$

Using the subspace classifier in the converse direction requires choosing a pullback for each predicate. We use **subfamilies**:

$$\frac{\Gamma, x : A \vdash \varphi : \mathbf{Prop}}{\Gamma \vdash \{x : A|\varphi\}} \qquad \{x : A|\varphi\}_\gamma \coloneqq \{a \in A_\gamma | \varphi a = \mathbf{true}\}$$
$$\mathcal{R}_{\{x:A|\varphi\}}{}^{\boldsymbol{v}} \coloneqq \left\{\alpha \in \mathcal{R}_A^{\boldsymbol{v}} \middle| \forall r : \mathbb{R}.(\alpha\, r) = \mathbf{true}\right\}$$

▶ **Proposition 29.** *The inclusion of a subfamily in a qbf is a fibred subspace embedding $\Gamma \vdash \mathbf{coerce}^\varphi \coloneqq \lambda a.a : \{x : A|\varphi\} \hookrightarrow A$. The subfamilies are **split**—this map is an identity:*

$$\Gamma \vdash \quad \mathsf{lift}\ \mathbf{coerce}^\varphi\,[\theta]\ \mathsf{along}\ \mathbf{coerce}^{\varphi[\theta]} \quad : \{x : A|\varphi\}\,[\theta] \to \{x : A\,[\theta]|\varphi\,[\theta]\}$$

The subfamily notation $\{x : A|\varphi\}$ can sometimes be awkward, and we will use the notation $\Gamma \vdash (x : A) \times \varphi$ for the same qbf. Also note the standard difference between $\Gamma \vdash \mathit{prf} : \exists x : A\varphi$ and $\Gamma \vdash M : \{x : A|\varphi\}$. The former merely asserts the predicate φ holds in each fibre. The latter asserts there is a measurable fibred map M that chooses a witness M in each fibre.

▶ **Example 30.** Let $[-\infty, \infty] \coloneqq \mathbb{R} \amalg \{\pm\infty\}$, the latter summand being the discrete qbs over two elements. Using subfamilies we can define qbfs for various intervals:

$$a, b : [-\infty, \infty] \vdash [a, b) \coloneqq \{r \in [-\infty, \infty]|a \le r < b\}$$

Let **Bool** $\cong \mathbb{1} \amalg \mathbb{1}$ be the discrete qbs over two elements $\{\mathbf{true}, \mathbf{false}\}$. It is a σ-field:

$$\neg : \mathbf{Bool} \to \mathbf{Bool} \qquad \bigvee_{i\in I}, \bigwedge_{i\in I} : \left(\bigtimes_{i\in I} \mathbf{Bool}\right) \to \mathbf{Bool} \qquad (I \text{ countable})$$

The qbf of **fibred events** is the fibred exponential into **Bool**. We also have a qbf for the **measurablity** of a proposition φ:

$$\frac{\Gamma \vdash A}{\Gamma \vdash \mathcal{B}_A \coloneqq A \to \mathbf{Bool}} \qquad \frac{\Gamma \vdash \varphi : \mathbf{Prop}}{\Gamma \vdash \mathbf{Decide}\,\varphi \coloneqq \varphi \amalg (\neg\varphi)} \qquad \begin{array}{l}\mathsf{Yes} \coloneqq \iota_1 : \quad \varphi \to \mathbf{Decide}\,\varphi \\ \mathsf{No} \ \coloneqq \iota_2 : \neg\varphi \to \mathbf{Decide}\,\varphi\end{array}$$

Booleans inject, but do not embed, into **Prop**, since we can match on Booleans, we can decide whether a Boolean is true:

$$\Gamma \vdash \mathsf{decide} := \lambda b.\mathbf{match}\, b\, \mathbf{with} \begin{Bmatrix} \mathbf{true} \Rightarrow \mathsf{Yes\,refl} \\ \mathbf{false} \Rightarrow \mathsf{No\,refl} \end{Bmatrix} : (b : \mathbf{Bool}) \to \mathbf{Decide}\, b$$

Recall [39] that a **Borel embedding** $e : X \hookleftarrow\!\!\!\!\rightarrow Y$ between two qbses is a subspace embedding $e : X \hookrightarrow Y$ such that the direct image $e[X] \subseteq Y$ is a (measurable) event in Y, i.e., its membership-testing function is in the space of events: $(\lambda y.y \overset{?}{\in} e[X]) \in \mathcal{B}_Y$. We similarly define a **fibred Borel embedding** $\Gamma \vdash e : A \hookleftarrow\!\!\!\!\rightarrow B$ between two qbfs to be a fibred subspace embedding $e : A \hookrightarrow B$, such that the following predicate is measurable:

$$\Gamma, y : B \vdash y \overset{?}{\in} e[A] := \exists a : A.b = e\, a : \mathbf{Prop}$$

i.e., there is some term $\Gamma, y : B \vdash \varphi? : \mathbf{Decide}(y \overset{?}{\in} e[A])$.

▶ **Example 31.** The **partial measurable-map classifier** for a qbf $\Gamma \vdash A$ is the fibred coproduct qbs $\Gamma \vdash A^{\perp} := A \amalg \{\perp\}$, cf. Vákár et al. [39] for the non-fibred concept. The **partial fres** are the qbf maps of the form $\Gamma \vdash f : \mathbb{R} \to A^{\perp}$. They are recombinations $f_\gamma = \left[E_\gamma.\iota_1 \circ g_\gamma, E^{\complement}_\gamma.\underline{\iota_2 \perp}\right]$ for a qbf map $\Gamma \vdash g : \{x \in A | x \in \mathrm{Dom}\,(f)\} \to A$ for:

$$\Gamma \vdash \mathrm{Dom}\,(f) := \lambda a.\mathbf{match}\, f a\, \mathbf{with} \begin{Bmatrix} \iota_1 x \Rightarrow \mathbf{true} \\ \iota_2 \_ \Rightarrow \mathbf{false} \end{Bmatrix} : \mathcal{B}_A$$

Since the qbs of natural numbers $\mathbb{N}$ is discrete and countable, the Ternalisation Thm. 22 lets us internalise typically inductive arguments. A common pattern we will use is that of **measurable selection/choice**. Given a predicate $\Gamma, x : A \vdash \varphi : \mathbf{Prop}$, we will sometimes want to internalise the fact that, when φ holds, there is a measurable function that constructs it. We will typically write these using the following quantifier notation: $\Gamma \vdash \mathsf{Ƨ}(x : A).\varphi$. All it means is that we have a term $\Gamma \vdash M : \{x : A | \varphi\}$.

▶ **Proposition 32.** *Let I be a countable set, $\Gamma, i : I \vdash \varphi : \mathbf{Prop}$ a proposition. If we have $\Gamma, i : I \vdash \varphi? : \mathbf{Bool}$ that lifts φ through the injection from the Booleans, there is a selection principle $\Gamma \vdash \mathsf{Ƨ}(i : I).\varphi$.*

The proof follows the analogous simply-typed selection [21], using the Ternalisation Thm. 22 and external induction on an external enumeration of I.

6 Fibred monads for statistics and probability

This section defines two fibred monads **D** and **P** for general distributions and probability distributions respectively. We first recall how these are defined as monads on the category **Qbs**, following more recent formulations [29, 39] of the original concepts [14, 9].

For distributions, let $\mathbb{W} = [0, \infty]$ be the qbs of **weights**, and recall the partial measurable-map classifier from Ex. 31. For any qbs X, the **integration** map integrates a measurable weight function using the law of a partial re in X w.r.t. the Lebesgue measure:

$$\int := \left(\lambda f.\lambda g.\int_{\mathrm{Dom}(f)} \mathrm{d}x (g \circ f)\right) : \underbrace{(\mathbb{R} \to X^{\perp})}_{=:\mathbf{R}X} \to \underbrace{(X \to \mathbb{W}) \to \mathbb{W}}_{=:\mathbf{K}X}$$

The integral is a higher order measurable function in **Qbs**, and we call its arguments from $\mathbf{R}X$ **randomisations** and from $\mathbf{K}X$ **integration operators**. The carrier of the **Qbs** distribution monad is $\underline{\mathbf{D}X} \coloneqq \int\left[\mathbb{R} \to X^{\perp}\right] \subseteq \underline{\mathbf{K}X}$ to be the image of the above map. It is tempting to define it as the subspace, but that will not give a monad with the correct properties (e.g., Fubini's theorem). Denoting the lifting of $\int$ along the subset embedding $i : \underline{\mathbf{D}X} \rightarrowtail \underline{\mathbf{K}X}$, in **Set**, by $q_{\mathbf{D}} : \underline{\mathbf{R}X} \to \underline{\mathbf{D}X}$, we inherit the randomisation metaphorology and the ability to evaluate events:

$$\mathcal{R}_{\mathbf{D}X} \coloneqq \{\alpha : \mathbb{R} \to \mathbf{D}X \mid \exists \beta \in \mathcal{R}_{\mathbf{R}X}.\alpha = q_{\mathbf{D}} \circ \beta\} \qquad \mu(E) \coloneqq i\, \mu(\lambda x. \overset{?}{\in} E)$$

Now we have a measurable function $q_{\mathbf{D}} : \mathbf{R}X \to \mathbf{D}X$ and the injection becomes an injective measurable function $i : \mathbf{D}X \rightarrowtail \mathbf{K}X$. We need further technical care in defining the unit and Kleisli extension operation for this monad. The **probability monad** is given by the subspaces of probability distributions $\mathbf{P}X \coloneqq \{\mu \in \underline{\mathbf{D}X} | \mu(X) = 1\}$.

Our construction proceeds similarly, using fibred randomisations integration operators:

$$\Gamma \vdash (\mathbf{R}A) \coloneqq (\mathbb{R} \to A \amalg \{\perp\}),\ (\mathbf{K}A) \coloneqq ((A \to \mathbb{W}) \to \mathbb{W})$$

Fortunately, several facts simplify the external description. Since the carriers of coproducts, quotients, and the fibred exponentials are given fibrewise, we have for qbf $\Gamma \vdash A$:

$$\underline{(\mathbf{R}A)}_{\gamma} = \mathbf{Qbs}(\mathbb{R}, A_{\gamma} \amalg \{\perp\}) = \underline{\mathbf{R}A_{\gamma}} \qquad \underline{(\mathbf{K}A)}_{\gamma} = \underline{\mathbf{K}(A_{\gamma})}$$

Since subspace fibres are also given fibrewise, we define the fibres of the fibred monads by fibrewise application of their qbs counterparts:

$$\underline{\mathbf{D}(A)}_{\gamma} \coloneqq \underline{\mathbf{D}(A_{\gamma})} \qquad \underline{\mathbf{P}(A)}_{\gamma} \coloneqq \left\{\mu \in \underline{\mathbf{D}(A)}_{\gamma} \middle| \mu(A_{\gamma}) = 1\right\} = \underline{\mathbf{P}(A_{\gamma})}$$

The map $q_{\mathbf{D}}$ is a family map: $q_{\mathbf{D}} : \underline{\mathbf{R}(A)}_{\gamma} = \underline{\mathbf{R}(A_{\gamma})} \to \underline{\mathbf{D}(A)}_{\gamma}$, so we can define the fibred metaphorology for distributions: $\mathcal{R}^{\boldsymbol{v}}_{\mathbf{D}A} \coloneqq \left\{\alpha \middle| \exists \beta \in \mathcal{R}^{\boldsymbol{v}}_{\mathbf{R}A}.\alpha = q_{\mathbf{D}} \overset{\boldsymbol{v}}{\circ} \beta\right\}$. This definition states that the fres over an re $\boldsymbol{v}$ must come from a **fibred** randomisation $\beta \in \mathbf{R}A$. While it is tempting to try and define $\mathbf{D}A$ as the subspace of $\mathbf{D}\left(\coprod_{\Gamma} A\right)$ comprising of those measures that are concentrated on one fibre, the resulting qbf is different, and does not form a fibred functor. The unit and Kleisli extension for $\mathbf{D}$ and $\mathbf{P}$ lift fibrewise.

▶ **Theorem 33.** *$\mathbf{D}$ and $\mathbf{P}$ define commutative fibred monads on* **Qbf**. *The fibres of the distribution monad satisfy Kock's axioms synthetic measure theory [22, 29], i.e., there are canonical isomorphisms $\Gamma \vdash \mathbf{D}\left(\coprod_{i\in I} A^i\right) \xrightarrow{\cong} \bigtimes_{i\in I} \mathbf{D}(A^i)$ for every countable set I.*

This theorem and the fact that $\mathbf{D}$ and $\mathbf{P}$ lift their qbs counterparts fibrewise let us use the existing probabilistic concepts in the dependently-typed setting.

▶ **Example 34.** The **Lebesgue measure** on an interval becomes a fibred measure via $a, b : \mathbb{R} \vdash \lambda_{[a,b]} : \mathbf{D}[a, b]$. More generally, every event yields a distribution on itself:

$$E : \mathcal{B}_{\mathbb{R}} \vdash \lambda_E \coloneqq q_{\mathbf{D}}\left(\lambda r.\mathbf{match}\, r \overset{?}{\in} E\, \mathbf{with}\ \{\mathbf{true} \Rightarrow \mathsf{lift} r; \mathbf{false} \Rightarrow \perp\}\right) : \mathbf{D}E$$

Thm. 33 implies we can re-scale fibred measures. The **fibred uniform** distribution is the normalised Lebesgue measure on a non-null event: $E : \mathcal{B}_{\mathbb{R}}, \lambda\, E > 0 \vdash \mathbf{U}_E \coloneqq \mathsf{lift}_{\frac{1}{\lambda E}} \cdot \lambda_E : \mathbf{P}E$.

In the discrete setting, Jacobs [18] indexes urn models by their number of balls.

▶ **Example 35.** A (fibred) measure is **absolutely continuous w.r.t.** another when there exists a **density** function that scales the latter to the former [29]:

$$\Gamma, \mu, \nu : \mathbf{D}X \vdash (\mu \ll \nu) \coloneqq \exists w : X \to \mathbb{W}. w \cdot \nu = \mu : \mathbf{Prop}$$

An **improper Bayesian statistical model** typically comprises of a pair of distributions in the dependent pair $\Gamma \vdash K, M : (\nu : \mathbf{D}X) \times \{\mu \in \mathbf{D}X | \mu \ll \nu\}$. The distribution K is called the **prior** and M the **posterior**. Any density function, i.e., a Radon-Nikodym derivative $\Gamma \vdash \frac{\mathrm{d}\mu}{\mathrm{d}\nu} : \mathbb{W} \to X$ is then a **likelihood function** of this model.

Part II
Dependently-typed probability theory

We now reap the fruit of our labour by studying random variable spaces. This is something we cannot do with measurable spaces due to classical impossibility results [16]. As we will see, the development with quasi Borel spaces and families is straightforward and natural, thanks to the availability of more spaces. The techniques we employ are standard in the topos theoretic and constructive mathematics communities, where in our setting 'constructive' means 'measurable'. We will prove that a set-theoretic function is measurable by finding an equivalent way to compute it by composing measurable building blocks. We will sometimes construct these building blocks by internalising some external construction, using the Ternalisation Thm. 22 and the convenience of our fibration. We proceed as follows. In Sec. 7 we define the relevant spaces of Lebesgue integrable random variables and some measurable operations on them. In Sec. 8 we review standard functional analysis concepts that we will use in our main theorem. The main innovation here is our ability to define proof-relevant counterparts to separability and orthonormality. In Sec. 9, we recall the conditional expectation and prove it inhabits a dependent-function space inexpressible by the classical theory.

7 Random variable spaces

A **(fibred) probability space** is a qbf $\Gamma \vdash \Omega$ equipped with a probability distribution $\Gamma \vdash \mu : \mathbf{P}\Omega$. A **random variable** (rv) in a qbf Ω is a function in $\Gamma \vdash \Omega \to [-\infty, \infty]$. We define the **signed fragments**, i.e., the **positive** and **negative** fragments of a random variable $\Gamma \vdash -^+, -^- : \Omega \to [0, \infty]$ by $f^+ \coloneqq \lambda\omega. \max\{f\,\omega, 0\}$ and $f^- \coloneqq \lambda\omega. \max\{-f\,\omega, 0\}$. We will often **clip** a rv to take values in a particular event:

$$\Gamma \vdash (\wedge) : (\Omega \to [-\infty, \infty]) \to \mathcal{B}_{[-\infty,\infty]} \to \Omega \to [-\infty, \infty]$$
$$\coloneqq \lambda f, E.\lambda\omega.\mathbf{match}\, E(f\,\omega)\,\mathbf{with}\,\{\mathbf{true} \Rightarrow f\,\omega; \mathbf{false} \Rightarrow 0\}$$

We extend Lebesgue integration to a partial function over rvs, in the usual way:

$$\Gamma \vdash \mathrm{Dom}\left(\int \mathrm{d}\mu\right) \coloneqq \left\{f \middle| \int \mathrm{d}\mu f^+ < \infty \vee \int \mathrm{d}\mu f^- < \infty\right\} : \mathcal{B}_{\Omega \to [-\infty,\infty]}$$
$$\int \mathrm{d} \coloneqq \lambda\mu, f. \left(\int \mathrm{d}\mu f^+\right) - \left(\int \mathrm{d}\mu f^-\right) : \mathbf{P}\Omega \to (\Omega \to [-\infty, \infty]) \rightharpoonup [-\infty, \infty]$$

An rv in a measure space is **Lebesgue integrable** when its integral is defined and finite:

$$\Gamma, f : \Omega \to [-\infty, \infty] \vdash \quad \mu\text{-integrable}\, f \coloneqq f \in \mathrm{Dom}\left(\int \mathrm{d}\mu\right) \wedge \int \mathrm{d}\mu\, |f| < \infty : \mathcal{B}_{\Omega \to [-\infty,\infty]}$$

We define the qbf of **Lebesgue spaces** by: $\dfrac{\Gamma \vdash \mu : \mathbf{P}\Omega \qquad \Gamma \vdash p : [1, \infty)}{\Gamma \vdash \mathcal{L}^p(\Omega, \mu) \coloneqq \{f : \Omega \to \mathbb{R} | \mu\text{-integrable } f^p\}}$.
This family makes an inherent use of type dependency, for both μ and p, although we will typically only use $p \coloneqq 1, 2$. Each Lebesgue space, i.e., fibre in this family, is a real vector space when equipped with the pointwise operations:

$$\begin{aligned} \Gamma \vdash (+) &\coloneqq \lambda f, g.\lambda\omega.(f\,\omega) + (g\,\omega) && : \mathcal{L}^p(\Omega, \mu)^2 \to \mathcal{L}^p(\Omega, \mu) \\ (\cdot) &\coloneqq \lambda r, f.\lambda\omega.r \cdot (f\,\omega) && : \mathbb{R} \times \mathcal{L}^p(\Omega, \mu) \to \mathcal{L}^p(\Omega, \mu) \end{aligned}$$

Each fibre also has a **seminorm**, satisfying the following non-negativity, triangle inequality, and homogeneity properties:

$$\Gamma \vdash \|-\|_p \coloneqq \lambda f. \sqrt[p]{\int \mathrm{d}\mu\, |f|^p} : \mathcal{L}^p(\Omega, \mu) \to \mathbb{W} \qquad \forall f. \|f\|_p \geq 0$$

$$\forall f, g. \|f + g\|_p \leq \|f\|_p + \|g\|_p \qquad \forall r, f. \|r \cdot f\|_p = |r| \cdot \|f\|_p$$

For the purpose of the sequel, an event $\Gamma \vdash E \in \mathcal{B}_{\mathcal{B}}$ is **μ-almost sure** when $\Gamma \vdash \mu E^{\complement} = 0$. When $\Gamma, \omega : \Omega \vdash \varphi : \mathbf{Bool}$, we will write $\mu(\mathrm{d}\omega)$-a.s. φ when the corresponding event $\lambda\omega.\varphi$ is μ-almost sure.

Each p seminorm is not a norm, because $\|f\|_p = 0$ need not imply $f = 0$, but it does imply that $\mu(\mathrm{d}\omega)$-a.s. $f\,\omega = 0$. In particular, we can measurably detect whether two integrable rvs are a.s. equal:

$$\Gamma, f, g : \mathcal{L}^p(\Omega, \mu) \vdash f \underset{\mu\text{-a.s.}}{\overset{?}{=}} g \coloneqq \|f - g\|_p \overset{?}{=} 0 : \mathbf{Decide}(f \underset{\mu\text{-a.s.}}{=} g)$$

We do not quotient the carriers of the Lebesgue space by a.s. equality. A related set is the Banach space $\mathrm{L}^1(X, \mu)$ that is studied in functional analysis, whose carrier is this quotient. The distinction between $\mathcal{L}^p(X, \mu)$ and $\mathrm{L}^p(X, \mu)$ is a standard distinction in probability theory and analysis. We often use the sets $\mathcal{L}(X, \mu)$ in probabilistic contexts since those allow (set-theoretic) evaluation everywhere and allow us to talk about, e.g., continuous-time-and-space processes being continuous functions. The space $\mathrm{L}(X, \mu)$ is a Banach space often studied in functional analysis, where indeed this kind of evaluation as meaningless unless it appears under an integral. This standard divide is covered in, e.g., Williams [41, p. 65]. We follow his proof of the existence of conditional expectation, which uses the Banach and Hilbert space structures of $\mathrm{L}^p(X, \mu)$ without quotienting, and so we will use $\mathcal{L}^p(X, \mu)$.

Each seminorm induces limits, called **convergence in p-mean**:

$$\Gamma, f : \mathbb{N} \to \mathcal{L}^p(\Omega, \mu), g : \mathcal{L}^p(\Omega, \mu) \vdash \quad (f_n \xrightarrow[n\to\infty]{\mathcal{L}^p} g) \coloneqq \forall \varepsilon > 0.\exists N : \mathbb{N}. \|f_n - g\|_p < \varepsilon : \mathbf{Prop}$$

This property is in fact measurable, since we can characterise by quantifying over the countable set of rational $\varepsilon : (0, \infty) \cap \mathbb{Q}$.

▶ **Proposition 36.** *Convergence in mean is measurable, and we can measurably find a limit:*

$$\Gamma \vdash \overset{p}{\lim} : (f : \mathbb{N} \to \mathcal{L}^p(\Omega, \mu)) \rightharpoonup \left\{g : \mathcal{L}^1(\Omega, \mu) \middle| f_n \xrightarrow[n\to\infty]{\mathcal{L}^p} g\right\}$$

Proof. The Lebesgue spaces are **complete**: a sequence converges iff it is Cauchy, and the latter property is measurable:

$$\Gamma, f : \mathbb{N} \to \mathcal{L}^p(\Omega, \mu) \vdash \quad \mathsf{Cauchy}\ f \coloneqq \forall \varepsilon : (0, \infty) \cap \mathbb{Q}.\exists N : \mathbb{N}.\forall m, n > N. \|f_n - f_m\| < \varepsilon$$

We can measurably define a limit for a Cauchy sequence by clipping any infinite values from its limit superior $\overset{p}{\lim}_{n\to\infty} f_n \coloneqq (\lambda\omega. \limsup_n (f_n \omega)) \wedge (-\infty, \infty)$. ◀

The following observation [40, proof of Thm. 21] underlies our development: Let X be a qbs and $M := \langle \underline{X}, \mathcal{B}_X \rangle$ the free measurable space over it. Then the sets of rvs in X and M coincide: $\mathbf{Qbs}(X, [-\infty, \infty]) = \mathbf{Meas}(M, [-\infty, \infty])$. All the operations we have mentioned here are measurable in their functional arguments. Qbs/qbfs are essential to state this fact.

8 Orthonormality and separability

The fibres of the Lebesgue spaces are dual to each other in the sense that bounded linear functionals of $\mathcal{L}^p$ can be represented up to a.s. equality by the rvs in $\mathcal{L}^q$ where $\frac{1}{p} + \frac{1}{q} = 1$. Evaluation of a functional at an rv is then given by:

$$\Gamma, \tfrac{1}{p} + \tfrac{1}{q} =::= 1 \vdash \quad \langle -, - \rangle : \lambda f, g. \int \mu(\mathrm{d}\omega)(f\,\omega) \cdot (g\,\omega) : \mathcal{L}^p(\Omega, \mu) \times \mathcal{L}^q(\Omega, \mu) \to \mathbb{R}$$

We will use only the self-dual case $p := 2 =: q$, where $\langle -, - \rangle$ becomes an inner-product except for the positivity axiom only implying a.s. equality. The space $\Gamma \vdash \mathcal{L}^2(\Omega, \mu)$ is then 'almost' a Hilbert space, with the same caveat. We will need a qbfs that capture properties of sets of rvs (orthonormality, density, etc.) and that will allow us to measurably search through those sets. We will replace those subsets with enumerations. We set up the concepts for these now.

Let $\mathbb{N}_\infty := \mathbb{N} \amalg \{\infty\}$ be the extended natural numbers. Define the discrete qbf of **countable indexing sets** to be the indices up to a given bound $k : \mathbb{N}_\infty \vdash \underline{k} := \{n \in \mathbb{N} | n < k\}$. The qbf of **countable enumerations** $\Gamma \vdash A$ is $\Gamma \vdash \mathbf{EnumIn} A := (k : \mathbb{N} \amalg \{\infty\}) \times (\underline{k} \to A)$. An enumeration of 2-integrable rvs is **orthonormal** when it enumerates an orthonormal subset:

$$\Gamma, \langle k, b \rangle : \mathbf{EnumIn}\mathcal{L}^2(\Omega, \mu) \vdash \mathsf{orthonormal}\,\langle k, f \rangle :=$$
$$\forall i < k.\, \langle b_i, b_i \rangle = 1 \wedge \forall j < k. j \neq i \implies \langle b_i, b_j \rangle = 0 : \mathbf{Prop}$$

We define the qbf of orthonormal enumerations, and show that orthonormality is measurable:

$$\Gamma \vdash \mathbf{Orthonormal}(\Omega, \mu) := (B : \mathbf{EnumIn}\mathcal{L}^2(\Omega, \mu)) \times \mathsf{orthonormal}\, B$$
$$\Gamma \vdash \mathsf{orthonormal}\,? : (B : \mathbf{EnumIn}\mathcal{L}^2(\Omega, \mu)) \to \mathbf{Decide}(\mathsf{orthonormal}\, B)$$

Orthonormality lets us detect measurably whether we can approximate a given rv via linear combination. Define the **orthogonal projection** onto the space of **linear approximations**:

$$\Gamma, \langle k, b \rangle : \mathbf{Orthonormal}(\Omega, \mu), f : \mathcal{L}^2(\Omega, \mu) \vdash \quad \pi_{\overline{\mathrm{Span}}_{i<k} b_i} f := \sum_{i<k} \langle f, b_i \rangle \cdot b_i : \mathcal{L}^2(\Omega, \mu)$$
$$f \overset{?}{\in} \overline{\mathrm{Span}}_{i<k} b_i := f \underset{\mu\text{-a.s.}}{\overset{?}{=}} \pi_{\overline{\mathrm{Span}}_{i<k} b_i} f : \mathbf{Decide}(f \overset{?}{\in} \overline{\mathrm{Span}}_{i<k} b_i)$$

A **measurable Schauder basis** for $\mathcal{L}^2$ is an orthonormal enumeration approximating all 2-integrable rvs f through linear combinations that a.s. converge to f:

$$\Gamma \vdash \mathbf{Schauder}(\Omega, \mu) := (\langle k, b \rangle : \mathbf{EnumIn}\mathcal{L}^2(\Omega, \mu)) \times \begin{pmatrix} \forall f : \mathcal{L}^2(\Omega, \mu). \exists r : \underline{k} \to \mathbb{R}. \\ \mu(\mathrm{d}\omega)\text{-a.s. } f\,\omega = \sum_{i<k} r_i \cdot b_i \omega \end{pmatrix}$$

Constructing a measurable Schauder basis is the main consequence of the regularity assumptions we will assume in order to construct a measurable version of the conditional expectation in Sec. 9. The first step is to build a Schauder basis out of any **dense** enumeration:

$$\Gamma, \langle k, b \rangle : \mathbf{EnumIn}\mathcal{L}^p(\Omega, \mu) \vdash \quad \overline{\langle k, b \rangle}^p := \left\{ f : \mathcal{L}^p(\Omega, \mu) \middle| \exists f : \mathbb{N} \to \underline{k}, f_n \xrightarrow[n \to \infty]{\mathcal{L}^p} f \right\}$$
$$\mathsf{dense}_p\, \langle k, b \rangle := \forall f \in \mathcal{L}^p(\Omega, \mu). f \in \overline{\langle k, b \rangle}^p \qquad : \mathbf{Prop}$$

▶ **Proposition 37.** *We can measurably construct a measurable Schauder basis from a* 2*-dense enumeration in* $\mathcal{L}^2$*, i.e.:* $\Gamma, B : \mathbf{EnumIn}\mathcal{L}^2(\Omega,\mu), \mathsf{dense}_2\, B \vdash \mathsf{GramSchmidt} : \mathbf{Schauder}(\Omega,\mu)$.

As the name for the term suggest, the proof consists of validating that we have a measurable version of the Gram-Schmidt process.

Next, we reduce the construction of a 2-dense enumeration to an equivalent property on events. A probability space is **measurably separable** when we have an enumeration that lets use approximate the probability of all events:

$$\Gamma, \langle k, E\rangle : \mathbf{EnumIn}\mathcal{B}_\Omega \vdash \mathsf{dense}_{\mathcal{B}}\,\langle k, E\rangle \coloneqq \forall F : \mathcal{B}_\Omega.\forall \varepsilon : (0,\infty).\exists i \in \underline{k}.\mu(F \setminus E_i) + \mu(E_i \setminus F) < \varepsilon : \mathbf{Prop}$$
$$\Gamma \vdash \mathbf{Separable}(\Omega,\mu) \coloneqq (B : \mathbf{EnumIn}\mathcal{B}_\Omega) \times \mathsf{dense}_{\mathcal{B}}\, B$$

Thanks to countable selection Prop. 32, we measurably search for an approximating event:

▶ **Proposition 38** (cf. Cohn [7, Prop. 3.4.5])**.** *In a separable space we can measurably enumerate dense sets:* $\Gamma, \mathbf{Separable}(\Omega,\mu) \vdash \mathsf{CohnProp} : (B : \mathbf{EnumIn}\mathcal{L}^p(\Omega,\mu)) \times \mathsf{dense}_p\, B$.

As the name suggests, the proof consists of reproducing the standard argument in the cited source as a measurable function. The final ingredient is sufficient conditions for separability:

▶ **Proposition 39** (cf. Cohn [7, Lemma 3.4.6])**.** *Enumerations of a probability space's events make it measurably separable:* $\Gamma, B : \mathbf{EnumIn}\mathcal{B}_\Omega, \sigma B = \mathcal{B}_\Omega \vdash \mathsf{CohnLemma} : \mathbf{Separable}(\Omega,\mu)$. *In particular, if* $\vdash S$ *is a* ***standard Borel space****, i.e., if* $\vdash S \hookleftarrow \mathbb{R}$*, then its probability spaces are separable* $\Gamma \vdash \mathsf{Regularity} : \mathbf{Separable}(S,\mu)$.

9 Conditional expectation

The conditional expectation formalises a relationship between statistics, i.e. rvs, in two different spaces connected by a measurable map. It may help intuition to name the various ingredients in the definition by their intended use.

As in the previous two sections, we have a sample space $\Gamma \vdash \mu : \mathbf{P}\Omega$. It typically describes both outcomes we can observe directly and outcomes we cannot. We also have an **observation space** Θ, which contains outcomes we can observe directly. Every **observation**, i.e., measurable map $H : \Omega \to \Theta$, induces a **law** $\Gamma \vdash \mu_H \coloneqq \mathbf{P}H\mu : \mathbf{P}\Theta$. We have an Ω-statistic—**latent variable** $\Gamma \vdash f : \mathcal{L}^1(\Omega,\mu)$— we want to study but may not be able to observe directly. And we have an observed statistic, $\Gamma \vdash g : \mathcal{L}^1(\Theta,\mu_H)$. We may want to involve the latter statistic g in other calculations, by integrating it against other rvs. The conditional expectation defines a formal relationship between such f and g that allows us to transport observable calculations involving the observed variable into facts concerning the latent variable. Formally, we say that g is a **conditional expectation** of f **given** H, when:

$$\Gamma \vdash (g = \mathbb{E}_\mu[f|H]) \coloneqq \forall \varphi \in \mathcal{L}^1(\Theta,\mu_H). \int \mathrm{d}\mu f \cdot (\varphi \circ H) = \int \mathrm{d}\mu_H g \cdot \varphi : \mathbf{Prop}$$

Textbook formulations of the conditional expectation [41, 20, 7] only involve one space, the sample space Ω. Instead of conditioning on an observation map $H : \Omega \to \Theta$, textbook formulations condition on a σ-field of observable events $\mathcal{G} \subseteq \mathcal{B}_\Omega$. Often, though not always, the defining condition for the conditional expectation is phrased only for $\varphi \coloneqq [- \in E]$ for some observable event $E \in \mathcal{G}$. We return to the textbook formulation by taking $\mathcal{G} \coloneqq H^{-1}[\mathcal{B}_\Theta]$.

In the classical, measure theoretic, setting, this reformulation is too limiting: we don't have enough spaces Θ and observations H around. But in qbs, where we have more of them, we prefer to use the observation-map formulation. For example, measure theoretic treatments of continuous-time processes make use of **filtrations**: monotone families of σ-fields $\mathcal{G}_t \subseteq \mathcal{B}_\Omega$ indexed by $t \in [0, T]$. The concept of a **continuous-time martingale** is then a measurable function $f : \Omega \times [0, T] \to \mathbb{R}$ that is compatible with the filtration such that $\mathbb{E}[f|\mathcal{G}_t] = f(-, t)$ for all t. The filtration serves a dual rôle in this definition, a measurability rôle that ensures the relevant conditional expectation is well-defined, and a causality rôle, specifying which events are known at each time $t \in [0, T]$.

Our formulation of the conditional expectation lets us treat martingales as random functions more directly. Take $\Omega \coloneqq ([0, T] \to \mathbb{R})$ and $t : [0, T] \vdash \Theta_t \coloneqq ([0, t) \to \mathbb{R})$ and:

$$t : [0, T] \vdash H_t \coloneqq \lambda f, x.f(\mathbf{coerce}^{[0,t] \subseteq [0,T]} x) : \Omega \to \Theta_t$$

We can then say that a process $\mu \in \mathbf{P}\Omega$ is a continuous time martingale when its value at time t is its expected value conditioned in its values up to time t: $t : [0, T] \vdash \mathbb{E}_{f \sim \mu}[f\, t|H_t] = f\, t$. In ordinary measure theory, there are no measurable spaces such as Ω and Θ, and so the definition of a continuous time martingale must use filterations instead.

We can relate the two notions of conditional expectation:

▶ **Proposition 40** (externalisation of the conditional expectation)**.** *Let $\Gamma \vdash \Omega, \Theta$ be qbfs, $\Gamma \vdash \mu : \mathbf{P}\Omega$ fibred probability measure, and $\Gamma \vdash H : \Omega \to \Theta$ fibred observation map. For every fibre $\gamma \in \underline{\Gamma}$, let M_γ and T_γ be the free measurable spaces over Ω_γ and Θ_γ, respectively, and $\mathcal{G}_\gamma \coloneqq H_\gamma^{-1}\left[\mathcal{B}_{\Theta_\gamma}\right] \subseteq \mathcal{B}_{\Omega_\gamma}$. For every rvs $\Gamma \vdash f : \mathcal{L}^1(\Omega, \mu\langle\rangle) =$ and $\Gamma \vdash g : \mathcal{L}^1(\Omega, (\mu\langle\rangle)_H)$, the qbf-notion of conditional expectation and the classical one coincide:*

$$\forall \gamma \in \underline{\Gamma}. g_\gamma = \mathbb{E}_{\mu_\gamma}[f|H] \iff \forall \gamma \in \underline{\Gamma}. g_\gamma \circ H_\gamma = \mathbb{E}_{\mu_\gamma}[f_\gamma|\mathcal{G}_\gamma]$$

Externalisation lets us derive the usual properties of the conditional expectation: a.s. uniqueness, linearity, etc. Constructive separability yields a measurable conditional expectation:

▶ **Theorem 41** (measurable conditional expectation)**.** *If the observation space Θ is separable, then we have a dependent function for the conditional expectation:*

$$\begin{aligned}\Gamma \vdash \mathbb{E}_{-}[-|-] : (\mu : \mathbf{P}\Omega) \to (H : \Omega \to \Theta) \to \mathbf{Separable}(\Theta, \mu_H) \\ \to (f : \mathcal{L}^1(\Omega, \mu)) \to \left\{g : \mathcal{L}^1(\Theta, \mu_H) \middle| g = \mathbb{E}_\mu[f|H]\right\}\end{aligned}$$

In particular, when $\vdash \Theta$ is a standard Borel space, we have:

$$\Gamma \vdash \mathbb{E}_{-}[-|-] : (\mu : \mathbf{P}\Omega) \to (H : \Omega \to \Theta) \to (f : \mathcal{L}^1(\Omega, \mu)) \to \left\{g : \mathcal{L}^1(\Theta, \mu_H) \middle| g = \mathbb{E}_\mu[f|H]\right\}$$

Proof. We follow a standard proof by Williams [41], first constructing conditional expectations for 2-integrable rvs, and then taking limits of conditional expectations $\lim_n \mathbb{E}_\mu[f^\pm \wedge [0, n]|H]$ for 1-integrable rvs. All of these constructions are measurable terms-in-contexts. The crux of the proof for 2-integrable rvs involves considering the closed vector subspaces $S_\gamma \coloneqq \left\{f \in \mathcal{L}^2(\Omega, \mu) \middle| \exists g \in \mathcal{L}^2(\Theta, \mu_H)\right\}$. Then, letting $\langle k, \varphi\rangle$ be the measurable Schauder basis for $\mathcal{L}^2(\Theta, \mu_H)$, show that $\sum_{i<k} \langle f, \varphi_i \circ H\rangle \cdot \varphi_i$ is a conditional expectation for f. ◀

10 Related work

Dependent types and probability. The last few years have seen proposals for semantic structure supporting dependent types. Smith [32] uses the category of **compactly generated Hausdorff spaces CGHaus** as a semantic setting for samplers from stochastic processes.

Since this category is Cartesian-closed, it admits non-dependent spaces of processes. Even though it is not locally Cartesian closed, and so the fibration does not admit all dependent-function spaces, Smith argues, and presents evidence, that it contains enough dependent-function spaces to reason about interesting examples, which include stochastic processes. As the morphisms in **CGHaus** are continuous functions, the semantics is equipped with a notion of weak convergence, something not readily available for quasi-Borel spaces. The fibred probability monad is given by the coproduct of fibres $\coprod_{\gamma\in\Gamma}\mathbf{P}X_\gamma$. Together with continuity, this means probability kernels cannot continuously move between fibres. We do not know whether that is a substantial restriction in practice.

St Clere Smithe [33] proposes a fibration for stochastic kernels, in order to develop semantics for probabilistic graphical models and factor graphs. The original version of this manuscript uses classical measure theory, but unfortunately contained a gap in a measurability proof. A revision [34] now uses quasi-Borel spaces, and the fibration it proposes is a Kleisli construction for a fibre-wise monad over the codomain fibration. This construction appears different from the Kleisli fibration for the fibred monad from Sec. 6. It seems close to a different construction on the codomain fibration that does not give a fibred monad, i.e., the functorial action does not preserve Cartesian morphisms.

Sheaves for probability and probabilistic semantics. Simpson's work [31, 30] on probability sheaves is close to ours: it constructs a topos where every object is a probability space and includes spaces of random variables. Indeed, quasi-Borel are the separated sheaves of a sheaf topos [37]. Probability sheaf toposes are convenient in supporting operations of interest. Internalisation and externalisation in these toposes is often less straightforward.

A related distinction between sheaf toposes and separated sheaf toposes is the notion of pointedness. A notable exception are toposes of group actions and continuous group actions, such as that of . [24] There, the site has a single object, recovering pointedness. Internalisation and externalisation is similarly smooth. Externalisation of exponentials/function-spaces in these contexts is less convenient because they are not simply the homsets.

Bidlingmaier [3] and Coquand and Spitters [8] develop a constructively-valid theory of integration, which is valid in any topos. Stassen et al. [35] and Marionneau et al. [25] incorporate probabilistic reasoning to guarded type theory [4], based on the topos of trees.

Semantics involving conditional expectation and random variables Kozen [23] analyses the martingale convergence theorem, which is formulated with conditional expectation, as a colimit construction. Much work on quantitative weakest precondition semantics for probabilistic programs uses random variables [2, 1]. Chen et al. [5, 6] study spaces of continuous time Markov processes for logical and metric characterisation of behavioural equivalence. Domain theory pioneered the use of random variable spaces as semantic domains [13], including recent work [10, 12].

11 Conclusion

We have presented a comprehensive characterisation of semantic structure for dependently-typed probability theory based on quasi-Borel spaces. We also demonstrated its use for organising relevant concepts from traditional probability theory, and that these semantic structures expose more measurability properties, compositionally, than standard treatments of the topic. We hope that that this work will lead to syntactic formalisms for dependently-typed probability theory, and would simplify the presentation of probabilistic semantics.

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A Grothendieck fibrations

We will use fibrations to structure our semantics categorically, mostly in phrasing the universal properties we characterise through concrete, set-theoretic, representations. We will only recapitulate the rudimentary theory, and refer to Jacobs's [17] textbook for a thorough treatment. To keep the exposition grounded in a running example, we will develop the **family** and **codomain** fibrations alongside fibrational concepts.

Let $p : \mathcal{A} \to \mathcal{B}$ be a functor. We call the category $\mathcal{B}$ the **base**, and the category $\mathcal{A}$ the **total** category. Given an object in the base $\Gamma \in \mathcal{B}$, the **fibre over** Γ is the not-necessarily-full subcategory $(\Gamma \vdash) \rightarrowtail \mathcal{A}$ consisting of all those objects A with $pA = \Gamma$ and morphisms $f : A \to B$ between them with $pf = \mathrm{id}_\Gamma$, which are called **vertical** morphisms. We will write $\Gamma \vdash A$ to indicate A is an object in the fibre over Γ, and $\Gamma \vdash f : A \to B$ for vertical morphisms. The established terminology is geometric, and typically depicted as in Fig. 9(a), which may help visualising these concepts.

For our running example, given a set Γ, a **Γ-indexed family of sets** $\Gamma \vdash A$, is an assignment of a set A_γ for every element $\gamma \in \Gamma$. A **family** is a pair $\langle \Gamma, A \rangle$ consisting of a set Γ, called the **indexing** set, and a family A over it. We will write $\Gamma \vdash A$ instead of $\langle \Gamma, A \rangle$. We call each set A_γ the **fibre** over γ. A **family map** $\langle \theta, f \rangle : (\Gamma \vdash A_-) \to (\Delta \vdash B_-)$ between families is a pair consisting of a function $\theta : \Gamma \to \Delta$ and an assignment of a function $f_\gamma : A_\gamma \to B_{\theta\gamma}$ between corresponding fibres for every index $\gamma \in \Gamma$. We will write $\theta \vdash f$ instead of $\langle \theta, f \rangle$. The identities are $\mathrm{id}_\Gamma \vdash \mathrm{id}_{A_-} : (\Gamma \vdash A) \to (\Gamma \vdash A)$ and we compose two maps:

$$(\Gamma \vdash A) \xrightarrow{\theta \vdash f} (\Delta \vdash B) \xrightarrow{\xi \vdash g} (\Theta \vdash C)$$

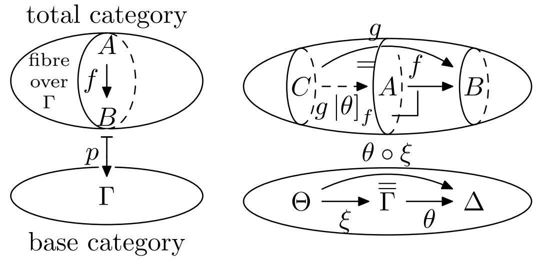

Figure 9 depicting (a) fibration and (b) Cartesian morphisms

as $(\xi \circ \theta) \vdash (g \overset{\theta}{\circ} f)$ where $(g \overset{\theta}{\circ} f)_\gamma \coloneqq g_{(\theta\,\gamma)} \circ f_\gamma$ is **dependent function composition**. Thus families and their maps form a category **Fam**. We get the situation in Fig. 9(a) by taking the category of sets and functions as the base $\mathcal{B} \coloneqq$ **Set**, families as the total category $\mathcal{A} \coloneqq$ **Fam**, and the first projection functor:

$$\text{index} : \Gamma \vdash A_{\_} \mapsto \Gamma \quad \theta \vdash f \mapsto \theta$$

The fibre over a set Γ is the sub-category of all families over Γ and the vertical maps have the form $(\text{id} \vdash f) : (\Gamma \vdash A) \to (\Gamma \vdash B)$. We will then write $\Gamma \vdash f : A \to B$ instead.

For every catgory $\mathcal{C}$, the category of **arrows** $\mathcal{C}^{\to}$ has as objects arrows $\overset{A}{\underset{\Gamma}{d\downarrow}}$ and as morphisms $(\theta, f) : \overset{A}{\underset{\Gamma}{d\downarrow}} \to \overset{B}{\underset{\Delta}{e\downarrow}}$ the evident commuting squares involving $\theta : \Gamma \to \Delta$ and $f : A \to B$. We get the situation in Fig. 9(a) by taking $\mathcal{C}$ as the base $\mathcal{B} \coloneqq \mathcal{C}$, arrows as the total category $\mathcal{A} \coloneqq \mathcal{C}^{\to}$, and the **codomain** functor sending

$$\text{cod} : \overset{A}{\underset{\Gamma}{d\downarrow}} \mapsto \Gamma \qquad \langle \theta, f\rangle \mapsto \theta$$

Given a functor $p : \mathcal{A} \to \mathcal{B}$ and consider an $\mathcal{A}$-morphism $f : A \to B$ **over** a $\mathcal{B}$-morphism $\theta : \Gamma \to \Delta$, i.e., $pf = \theta$. We say that f is **Cartesian** when for every $\mathcal{B}$-morphism $\xi : \Theta \to \Gamma$ and morphism $g : C \to B$ over the composition $\theta \circ \xi : \Theta \to \Delta$, there is a unique $\mathcal{A}$-morphism $g\,|\theta|_f : C \to A$ lifting g along f, i.e. $g = f \circ g\,|\theta|_f$. Fig. 9(b) depicts this situation. A family map $(\theta \vdash f) : (\Gamma \vdash A) \to (\Delta \vdash B)$ is Cartesian iff each component is bijective: $f_\gamma : A_\gamma \xrightarrow{\cong} B_{\theta\gamma}$. A commuting square in $\mathcal{C}^{\to}$ is Cartesian iff it is a pullback square.

A functor $p : \mathcal{A} \to \mathcal{B}$ is a **fibration** when there is, for every $\mathcal{B}$-morphism $\theta : \Gamma \to \Delta$ and every $\mathcal{A}$-object $\Delta \vdash B$, a Cartesian $\mathcal{A}$-morphism $(\theta \vdash f) : (\Gamma \vdash A) \to (\Delta \vdash B)$ over θ. A **cleavage** for a fibration is a choice of Cartesian morphisms:

$$\langle \theta, B\rangle \mapsto (\text{strengthen}^{\theta,B} : (\Gamma \vdash B\,[\theta]) \to (\Delta \vdash B))$$

for every $\theta : \Gamma \to \Delta$ and $\Delta \vdash B$. Each cleavage determines, for each $\theta : \Gamma \to \Delta$, the **reindexing functor** $-\,[\theta] : (\Delta \vdash) \to (\Gamma \vdash)$ through the universal property of the chosen Cartesian morphisms over θ. A **split** fibration is a fibration equipped with a cleavage whose induced reindexing functor assignment $\theta \mapsto -\,[\theta]$ is functorial:

$$-\,[\text{id}_\Gamma] = \text{Id}_{(\Gamma\vdash)} \qquad -\,[\theta \circ \xi] = (-\,[\theta])\,[\xi] \qquad\qquad (\Gamma \xrightarrow{\xi} \Delta \xrightarrow{\theta} \Theta)$$

Our two running examples have many cleavages each. The family fibration has a canonical cleavage, making the fibration split:

$$B\,[\theta] \coloneqq B_{\theta\,\_} \qquad \text{strengthen}^{\theta,B}_\gamma : B\,[\theta]_\gamma = B_{\theta\,\gamma}$$

$$
\begin{array}{ll}
\mathrm{dep}\_ : \mathbf{Fam} \to \mathbf{Set}^{\to} & \left(\mathrm{cod} \vdash -^{-1}[-]\right) : \mathbf{Set}^{\to} \to \mathbf{Fam} \\
\mathrm{dep}_{\Gamma \vdash A\_} \coloneqq \begin{smallmatrix} \coprod_{\Gamma} A\_ \\ \mathrm{dep} := \lambda \langle \gamma, a \rangle . \gamma \downarrow \\ \Gamma \end{smallmatrix} & \left(\mathrm{cod} \vdash -^{-1}[-]\right)\left(\begin{smallmatrix} A \\ d\downarrow \\ \Gamma \end{smallmatrix} \right) \coloneqq \left(\Gamma \vdash d^{-1}[-]\right) \\
\mathrm{dep}_{\left(\begin{smallmatrix} \Gamma \vdash A\_ \\ \theta \vdash f\_ \downarrow \\ \Delta \vdash A\_ \end{smallmatrix} \right)} \coloneqq \langle \theta, \coprod_{\theta} f\_ \rangle & \left(\mathrm{cod} \vdash -^{-1}[-]\right)\left(\begin{smallmatrix} A \xrightarrow{d} \Gamma \\ \langle \theta, f \rangle \downarrow \\ B \xrightarrow{e} \Delta \end{smallmatrix} \right)_{\gamma} \\
 & \coloneqq \left(\theta \vdash f|_{d^{-1}[\gamma]}\right) \\
\mathrm{id} \vdash \left(\eta_{\Gamma \vdash A\_} : A\_ \xrightarrow{\cong} (\mathrm{dep}_{\Gamma \vdash A})^{-1}[-]\right) & \eta \coloneqq \langle \lambda a . \langle \gamma, a \rangle \rangle_{\gamma \in \Gamma} \\
\left(\mathrm{id}, \varepsilon_{\begin{smallmatrix} A \\ d\downarrow \\ \Gamma \end{smallmatrix}} : \begin{smallmatrix} \coprod_{\Gamma} d^{-1}[-] \\ \mathrm{dep}\downarrow \\ \Gamma \end{smallmatrix} \to \begin{smallmatrix} A \\ d\downarrow \\ \Gamma \end{smallmatrix} \right) & \varepsilon \coloneqq \lambda \langle \gamma, a \rangle . a
\end{array}
$$

Figure 10 codomain and family fibration equivalence

Concrete categories typically have canonical choices for pullbacks, thus equipping the codomain fibration with cleavages. For **Set**:

$$\left(\begin{smallmatrix} B \\ d\downarrow \\ \Delta \end{smallmatrix} \right)[\theta] \coloneqq \{\langle \gamma, b \rangle \in \Gamma \times B | \theta\gamma = db\} \quad \mathrm{strengthen}^{\theta, d} \coloneqq \lambda \langle \gamma, b \rangle . b$$

This cleavage does not make the codomain fibration split.

A **(vertically) fibred** functor $F : p \to q$ between base-sharing fibrations $\mathcal{A} \xrightarrow{p} \mathcal{B} \xleftarrow{q} \mathcal{C}$ is a functor $F : \mathcal{A} \to \mathcal{C}$ that lifts p along q, i.e.: $p = q \circ F$, that moreover sends p-Cartesian morphisms to q-Cartesian morphisms. When p and q are split fibrations, a **split** functor is a fibred functor that sends the chosen Cartesian morphisms $\mathrm{strengthen}^{\theta, B}$ in p to the chosen Cartesian morphisms $\mathrm{strengthen}^{\theta, FB}$. A **(vertically) fibred** natural transformation between fibred functors, whether they are split or merely fibred, $p \overset{F}{\underset{G}{\Downarrow\alpha}} q$ is a natural transformation $\alpha : F \Rightarrow G$ whose components are vertical morphisms $\Gamma \vdash \alpha_A : FA \to GA$. Fibrations, fibred functors, and fibred natural transformations collect into a 2-category, allowing us to talk about fibred adjoint equivalences between fibration. One can similarly define not-necessarily-vertical split/fibred functors between any two split/mere fibrations as a pair of functors [17, Def. 1.7.7].

Returning to our running example, the family and codomain fibration over **Set** are fibred adjoint equivalent through these constructions. First, the **Grothendieck construction** $\coprod : \mathbf{Fam} \to \mathbf{Set}$:

$$(\Gamma \vdash A) \mapsto \coprod_{\Gamma} A \coloneqq \{\langle \gamma, a \rangle | \gamma \in \Gamma, a \in A_\gamma\} \qquad \text{(on families)}$$

$$(\theta \vdash f) \mapsto \coprod_{\theta} f \coloneqq \lambda \langle \gamma, a \rangle . \langle \theta\, a, f_\gamma a \rangle \qquad \text{(on maps)}$$

Figure 10 presents the fibred equivalence, projecting the index from the Grothendieck construction and by taking the preimage family.