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*ALGEBRA VALUED FUNCTORS IN GENERAL AND TENSOR PRODUCTS IN PARTICULAR**

BY

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In the recent past people have looked at non-standard models of algebraic theories. Today I wish to look at models of algebraic theories in non-standard categories. I am by no means the first to do so.

I began with a simple question: What is an algebra in the standard category, that is, the category of sets? My answer must be formulated in category predicates — objects and maps, but no elements. When so formulated it can be applied to an arbitrary category.

Let us start with a special case and formulate the axioms for a group. A *group* is a set A together with three maps: $m: A\times A \to A$, $n: A \to A$, $e: A \to A$ such that

1) associativity:

$$A\times A\times A \xrightarrow{1\times m} A\times A \xrightarrow{m} A = A\times A\times A \xrightarrow{m\times 1} A\times A \xrightarrow{m} A,$$

2) identity constant:

$$A\times A \xrightarrow{1\times e} A\times A \xrightarrow{m} A = A \xrightarrow{1} A,$$

3) inverses

$$A\times A \xrightarrow{1\times n} A\times A \xrightarrow{m} A = A \xrightarrow{e} A$$

(1 denotes the identity map).

We are not finished. The map e is supposed to be a constant map. That is, we must adjoin the equation $e(x) = e(y)$. If we understand $p_i(x_1, x_2) = x_i$, then

4) $A\times A \xrightarrow{p_1} A \xrightarrow{e} A = A\times A \xrightarrow{p_2} A \xrightarrow{e} A$.

Now if we know what $A\times A$ and the two maps p_1, p_2 mean in an arbitrary category we should know what we mean by a group in an

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arbitrary category. The categorical definition of product can be found in many places starting with Eilenberg-MacLane [1]. It can also be found in *Abelian Categories*, a book I feel obliged to mention for a number of reasons, among which is the fact that publisher gave me an advance [1].

For a number of well-known categories the groups therein have well-known names:

CATEGORY	GROUPS THEREIN
Topological Spaces	Topological Groups
Algebraic Manifolds	Algebraic Groups
Differential Manifolds	Lie Groups
Topological Spaces and Homotopy Classes of Maps	H-Spaces

One of the chief virtues of a group A in a category $\mathscr{A}$ is that for an arbitrary object $B \in \mathscr{A}$ the set of maps from B to A, which I denote by (B, A), is a group in the category of sets. To wit:

$$(B, A)\times (B, A) \xrightarrow{d} (B, A\times A) \xrightarrow{(B,m)} (B, A)$$

is its multiplication, where d is the one-to-one correspondence that arises from the definition of product, and (B, m) is the function obtained by composing with m. If B is held fixed, $(B, -)$ is a product preserving functor and perforce preserves all the axiomatic equations. If we hold A fixed, $(-, A)$ becomes a contravariant functor with values in the category of groups and homomorphisms. It is an example of what we shall call a *representable algebra valued functor*.

I think that the first deep theorem about functors was the Eilenberg-MacLane discovery [2] of the $K(\pi, n)$ spaces. They showed that the n-th cohomology functor with coefficients in a group π is naturally equivalent to the functor $\big(-, K(\pi, n)\big)$ where $K(\pi, n)$ is a group in the category of spaces and homotopy classes of maps.

Now let us generalize. Let $\mathbb{T}$ be an equational algebraic theory. There are a number of ways of formalizing that notion. I pick the most primitive: $\mathbb{T}$ is an indexed family of operator symbols $\{f_i\}$ together with a family of non-negative integers $\{v_i\}$ indexed over the same set, together with a family of equations relating the f_i's each of which looks in the equation as if it were a function on v_i arguments.

Given a theory $\mathbb{T}$ and a category $\mathscr{A}$ we shall say that a *$\mathbb{T}$-algebra* in $\mathscr{A}$ is an object $A \in \mathscr{A}$ together with a *$\mathbb{T}$-structure*, that is for each $f_i \in \mathbb{T}$ there is assigned a map $\bar{f}_i : \Pi_{v_i} A \to A$ and the collection is such that

[1] For related problems see the dedication in that book, [4], p. minus 3. (*Note of the Editors.*)

the equations of $\mathbb{T}$, when interpreted as below, are true. To interpret an equation it suffices to interpret an n-ary expression in the f's as a map in $\mathscr{A}$. The interpretation of the expressions follows from the recursive rules:

0) The interpretation of f_i is $\bar{f}_i$.

1) If g is an n-ary expression, and $g(x_1, x_2, \ldots, x_n) = x_j$, then we interpret g as $p_j : \Pi_n A \to A$, the j-th projection.

2) If $g(x_1, \ldots, x_n) = f_i(h_1(x_1, \ldots, x_n), \ldots, h_{v_i}(x_1, \ldots, x_n))$ and if we have already interpreted the h's as maps $\bar{h}_j : \Pi_n A \to A$, then we interpret g as the composition

$$\Pi_n A \xrightarrow{\Pi h_j} \Pi_n A \xrightarrow{\bar{f}_i} A.$$

I shall apologize for this primitive notion oft heory and the attendant cumbersome notion of $\mathbb{T}$-structure. I am in a quandry. The elegant definition is that used by Lawvere in his dissertation [6], in which theories are categories of a particular kind, and the algebras in $\mathscr{A}$ are particular kinds of functors from the theory into $\mathscr{A}$ (and the homomorphisms between the algebras are just natural transformations between the functors). I reject his formulation for expository reasons peculiar to our times — the language of categories is still too new. (Lawvere's definitions and some of his remarkable theorems are described in his announcement [7].)

Contravariant representable functors. If $A \in \mathscr{A}$ has a $\mathbb{T}$-structure, then for every $B \in \mathscr{A}$, (B, A) is a $\mathbb{T}$-algebra (in the category of sets) and the functor $(-, A)$ may be interpreted as a $\mathbb{T}$-algebra valued functor. In particular, a map $B \to B' \in \mathscr{A}$ induces a homomorphism $(B', A) \to (B, A)$.

Let $\mathscr{S}$ be the category of sets and $\mathscr{S}^{\mathbb{T}}$ the category of $\mathbb{T}$-algebras and homomorphisms in $\mathscr{S}$ (in other words, the ordinary notion of the category of $\mathbb{T}$-algebras). We shall say that $(-, A) : \mathscr{A} \to \mathscr{S}^{\mathbb{T}}$ is a contravariant representable functor, represented by the $\mathbb{T}$-algebra $A \in \mathscr{A}$, or more compactly, by $A \in \mathscr{A}^{\mathbb{T}}$, where $\mathscr{A}^{\mathbb{T}}$ is the category of $\mathbb{T}$-algebras and homomorphisms in $\mathscr{A}$.

In the language of [4], let $\mathscr{A}$ be a complete category and $F : \mathscr{A} \to \mathscr{S}^{\mathbb{T}}$ a contravariant functor. F is representable if and only if it has an adjoint on the right. I shall indicate the proof in a latter section. But for the moment, allow me to translate the above assertion together with the content of the Special Adjoint Functor Theorem of [4]:

THEOREM 1. *Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be equational algebraic theories and F : $\mathscr{S}^{\mathbb{T}_1} \to \mathscr{S}^{\mathbb{T}_2}$ a contravariant functor. F is representable if and only if*

1) For every set $\{B_i\}$ in $\mathscr{S}^{\mathbb{T}_1}$ there is a natural isomorphism $F(\sum B_i) \to \prod F(B_i)$ (where $\sum$ means free sum and $\prod$ means product).

2) *Let* $f: A \to A' \in \mathscr{S}^{T_1}$ *be an onto homomorphism,* $K = \{\langle a_1, a_2\rangle | f(a_1) = f(a_2)\}$, *and* $p_i: K \to A$ *be defined by* $p_i(a_i, a_2) = a_i$.

Then $F(f): F(A') \to F(A)$ *is one-to-one and the image of* $F(f)$ *is* $\{x \in F(A) | F(p_1)x = F(p_2)x\}$.

In fact, there are not too many familiar examples of contravariant representable functors. It will be the covariant case that is of most interest. But as an example of the power of such a theorem let me indicate how it can be used to construct injective modules, a task first performed by Rheinhold Baer using nothing but his own ingenuity and the ordinal numbers.

Injective algebras and cogenerators. Given a theory T and an algebra $A \in \mathscr{S}^T$, A is *injective* if for every $B \in \mathscr{S}^T$ and subalgebra $B' \subset B$ and homomorphism $f: B' \to A$ there is an extension $\bar{f}: B \to A$ such that $\bar{f}|B' = f$. In the case that T is the theory of left modules over a ring R, this definition specializes to the usual.

A is a *cogenerator* if for every $B \in \mathscr{S}^T$ and pair of distinct $x, y \in B$ there exists a homomorphism $f: B \to A$ such that $f(x) \neq f(y)$, unless B has a single element in which case we require $(B, A) \neq \emptyset$. Note that if A is a cogenerator, then we obtain a one-to-one map $B \to \Pi_I A$ for sufficiently large I (in particular for $I = (B, A)$). If $\mathscr{S}^T$ contains an injective cogenerator, then every T-algebra may be embedded in an injective algebra because products of injectives are easily seen to be injective.

The subject becomes a subject about functors once we make the following two observations:

A is injective if and only if $(-, A)$ *carries one-to-one maps into onto maps.*

A is a cogenerator if and only if $(-, A)$ *is a one-to-one functor.*

Now let T_1 be the theory of left R modules, T_2 the theory of abelian groups. For Q the group of rationals, $Z \subset Q$ the group of integers, it is the case that Q/Z is a divisable group and hence by Zorn's lemma, an injective object in $\mathscr{S}^{T_2}$. Moreover Q/Z is a cogenerator in $\mathscr{S}^{T_2}$. Consider the functor $\mathscr{S}^{T_1} \to \mathscr{S}^{T_2}$ which forgets the module structure and remembers only the underlying group structure. Such a functor is called *forgetful.* It is a covariant functor that preserves the hypotheses of two conditions in Theorem 1. Consider the functor $(-, Q/Z): \mathscr{S}^{T_2} \to \mathscr{S}^{T_1}$. Being representable, it carries the hypotheses of the two conditions into their conclusions. Hence the composition $\mathscr{S}^{T_1} \to \mathscr{S}^{T_2}$ satisfies the two conditions and is representable, say by $A \in \mathscr{S}^{T_1}$. But the two functors we are composing are both one-to-one and hence A is a cogenerator. The forgetful functor carries one-to-one maps into one-to-one maps and $(-, Q/Z)$ carries them into onto maps, hence A is injective.

Constant maps and zero-ary operations. I have so far tried to ignore one problem: what is a 0'ary operation on $A \in \mathscr{A}$. We can of course define 0'ary operations by using unary operations and by adjoining the revelant equation, namely that which will become interpreted as

1) $A \times A \xrightarrow{p_1} A \xrightarrow{f} A = A \times A \xrightarrow{p_2} A \xrightarrow{f} A$.

That equation is equivalent to

2) For all $g_1 : B \to A$, $g_2 : B \to A \in \mathscr{A}$

$$B \xrightarrow{g_1} A \xrightarrow{f} A = B \xrightarrow{g_2} A \xrightarrow{f} A.$$

Statement 2) seems best to generalize the notion of constant function to the notion of constant operator in a category.

We are faced, however, with the anomaly that this solution to the problem of 0'ary operations makes the empty set a model (in sets) of every algebraic theory. To find the alternate solution we apply what by now has become a standard method. Whatever a generalized 0'ary operation on A is, it should induce an ordinary 0'ary operation on (B, A). Hence by fiat we declare that a 0'ary operation on A is a collection $\{f_B \in (B, A)\}_{B \in \mathscr{A}}$ such that for every $B \to B' \in \mathscr{A}$, the induced map $(B', A) \to (B, A)$ carries $f_{B'}$ into f_B (and hence will be a homomorphism). Stated another way the diagram

$$\begin{tikzcd} B \arrow[d] \arrow[dr, "f_B"] & \\ B' \arrow[r, "f_{B'}"'] & A \end{tikzcd}$$

commutes regardless of the choice of $B \to B'$.

Now f_A will be a constant map as just defined and furthermore $B \xrightarrow{f_B} A = B \to A \xrightarrow{f_A} A$ for arbitrary choice of $B \to A$. But there is an extensional difference between our two definitions: whereas a generalized 0'ary operation determines a constant map which in turn determines the 0'ary operation the converse is false.

If $(B, A) \neq \emptyset$ for all $B \in \mathscr{A}$, then there is a one-to-one correspondence between the constant maps on A and the 0'ary operations on A.

If $(B, A) = \emptyset$ some $B \in \mathscr{A}$, then there are no 0'ary operations on A.

Tensor products of algebraic theories. Let T_1 and T_2 be theories, $\mathscr{S}$ the category of sets and consider $(\mathscr{S}^{T_1})^{T_2}$ which may be described as the category of T_2-algebras in the category of T_1-algebras. Such an algebra is a set A with a T_1-structure and a T_2-structure and such that each T_2 operation is a T_1-homomorphism. That is, given $f_n \in T_1$, $g_m \in T_2$ and a set of points in A $\{x_{ij}\}_{j=1,\dots,v_m}^{i=1,\dots,v_n}$ we obtain the equation

$$\text{(b)} \qquad f_n\big(g_m(x_{11},\dots,x_{1v_m}),\dots,g_m(x_{v_n1},\dots,x_{v_nv_m})\big) = g_m\big(f_n(x_{11},\dots,x_{v_n1}),\dots,f_n(x_{1v_m},\dots,x_{v_nv_m})\big).$$

The set of such equations is symmetric in f and g. All the $\mathbb{T}_1$-operators are $\mathbb{T}_2$-homomorphisms: $(\mathscr{S}^{\mathbb{T}_1})^{\mathbb{T}_2}$ is equivalent to $(\mathscr{S}^{\mathbb{T}_2})^{\mathbb{T}_1}$.

We can do better. There is a theory $\mathbb{T}$ such that $\mathscr{S}^{\mathbb{T}}$ is equivalent to $(\mathscr{S}^{\mathbb{T}_1})^{\mathbb{T}_2}$. We shall call such theory the *tensor product* of the two theories and denote it by $\mathbb{T}_1\otimes\mathbb{T}_2$.

The operators of $\mathbb{T}_1\otimes\mathbb{T}_2$ are obtained by taking the union, in the disjoint sense, of the operations from $\mathbb{T}_1$ and $\mathbb{T}_2$. The equations of $\mathbb{T}_1\otimes\mathbb{T}_2$ are obtained by adjoining to the equations from $\mathbb{T}_1$ and $\mathbb{T}_2$ all the equations of the form (b), one for each pair of operators from $\mathbb{T}_1$ and $\mathbb{T}_2$. This construction is universal: for every category $\mathscr{A}$, $\mathscr{A}^{\mathbb{T}_1\otimes\mathbb{T}_2}$ is equivalent to $(\mathscr{A}^{\mathbb{T}_1})^{\mathbb{T}_2}$.

The tensor operation on theories is commutative and associative.

With mild restriction on $\mathbb{T}_1$ and $\mathbb{T}_2$ it turns out that $\mathbb{T}_1\otimes\mathbb{T}_2$ collapses to a rather simple type of theory.

PROPOSITION. *If both $\mathbb{T}_1$ and $\mathbb{T}_2$ have constants, then $\mathbb{T}_1\otimes\mathbb{T}_2$ has a unique constant.*

Proof. If f is a $\mathbb{T}_1$-constant, g a $\mathbb{T}_2$-constant, then the tensor equation yields $fg = gf$ and hence $f = g$.

PROPOSITION. *If both $\mathbb{T}_1$ and $\mathbb{T}_2$ have binary operators each with a two-sided neutral constant (just as in the Jónsson-Tarski theorem [5]), then $\mathbb{T}_1\otimes\mathbb{T}_2$ is the theory of modules over a half-ring (a ring without subtraction).*

Proof. Let $\underset{i}{+}$ be the hypothesized operation coming from $\mathbb{T}_i$. The two constants are one, so if we call it 0, then $x\underset{i}{+}0 = x = 0\underset{i}{+}x$. The tensor equation gives

$$(w\underset{1}{+}x)\underset{2}{+}(y\underset{1}{+}z) = (w\underset{2}{+}y)\underset{1}{+}(x\underset{2}{+}z).$$

If we let $x = y = 0$, then $w\underset{1}{+}z = w\underset{2}{+}z$.

Erasing the subscripts: $(w+x)+(y+z) = (w+y)+(x+z)$, which equation implies the associativity and commutativity of $+$ (let $x=0$, then let $w = z = 0$).

Every operation is a sum of unary operations, because every operation must be a $+$ homomorphism and hence

$$f(x_1,\dots,x_n) = f(x_1,0,\dots,0)+f(0,x_2,0,\dots,0)+\dots+f(0,\dots,0,x_n).$$

The set of unary expressions together with $+$ generate the theory. Let U be the set of unary expressions. It is closed under $+$ and composition and becomes a half-ring with 0 and 1.

A $\mathbb{T}_1\otimes\mathbb{T}_2$-algebra is a commutative semi-group with 0, on which U operates, q. e. d.

And if $\mathbb{T}_1$ and $\mathbb{T}_2$ started out being the theories of modules over rings R_1 and R_2, respectively, then $\mathbb{T}_1\otimes\mathbb{T}_2$ is the theory of modules over $R_1\otimes R_2$ and $R_1\otimes R_2$ may be identified as the ring of unary expressions in $\mathbb{T}_1\otimes\mathbb{T}_2$.

The next propositions may be interpreted as saying that the theories of modules over half-rings form an ideal in the class of theories.

PROPOSITION. *If $\mathbb{T}_1$ is the theory of modules over a half-ring R, then for any $\mathbb{T}_2$, there is a half-ring R' such that $\mathbb{T}_1\otimes\mathbb{T}_2$ is the theory of modules over R'.*

Proof. Let $+$ be the additive operator from $\mathbb{T}_1$.

Every expression in $\mathbb{T}_1\otimes\mathbb{T}_2$ is a homomorphism with respect to $+$, and as in the proof of the last proposition, every expression is a sum of unary expressions. Hence the set of unary expressions is a half-ring and its modules are the models of $\mathbb{T}_1\oplus\mathbb{T}_2$, q. e. d.

Some other examples of tensor products of theories:

If $\mathbb{T}_1$ is the theory of modules over R and $\mathbb{T}_2$ is the theory with a single unary operator X and no equations, then $\mathbb{T}_1\otimes\mathbb{T}_2$ is the theory of modules over the polynomial ring $R[X]$.

If $\mathbb{T}_1$ is as above and $\mathbb{T}_2$ is the "discrete" theory with unary operators $X_1, \ldots, X_n$ and no equations, then $\mathbb{T}_1\otimes\mathbb{T}_2$ is the theory of modules over $R[X_1, \ldots, X_n]$ where it is understood that the X's do not commute.

If $\mathbb{T}_1$ is as above and $\mathbb{T}_2$ is the commutative discrete theory, that which has unary operators $X_1, \ldots, X_n$ and the equations $\{X_iX_j = X_jX_i\}$, then $\mathbb{T}_1\otimes\mathbb{T}_2$ is the theory of modules over $R[X_1, \ldots, X_n]$, where now the X's do commute.

The last $\mathbb{T}_2$, the commutative discrete theory, is the n-fold tensor product of the simplest non-empty discrete theory.

In general, if $\mathbb{T}_1$ is as above and $\mathbb{T}_2$ is any theory all of whose operators are unary, then $\mathbb{T}_1\otimes\mathbb{T}_2$ is the theory of modules over the R-algebra generated by the operators of $\mathbb{T}_2$ reduced by the equations of $\mathbb{T}_2$. This condition on $\mathbb{T}_2$ is equivalent to saying that there is a semi-group $\mathscr{V}_2$ and that $\mathscr{V}_2$-algebras are sets on which $\mathscr{V}_2$ operates. $\mathbb{T}_1\otimes\mathbb{T}_2$ is the semi-group algebra $R[\mathscr{V}_2]$.

If $\mathbb{T}_3$ is the theory of sets with a semi-group $\mathscr{V}_3$ acting on them, then $\mathbb{T}_2\otimes\mathbb{T}_3$ corresponds to $\mathscr{V}_2\times\mathscr{V}_3$. The associativity and commutativity of tensor products thus yields that $R[\mathscr{V}_2]\otimes R'[\mathscr{V}_3] \simeq R\otimes R'[\mathscr{V}_2\times\mathscr{V}_3]$.

Co-algebras. Let $\mathcal{T}$ be an algebraic theory. A $\mathcal{T}$-co-algebra in $\mathscr{A}$ may be defined as a $\mathcal{T}$-algebra in $\mathscr{A}^*$, the dual category. It is useful however, to translate back to $\mathscr{A}$: a $\mathcal{T}$-co-algebra in $\mathscr{A}$ is an object $A \in \mathscr{A}$ together with a *$\mathcal{T}$-co-structure*, that is a collection of maps $\{\bar{f}_i : A \to \Sigma_{v_i} A\}_i$ such that for every equation $g(x_i, \dots, x_n) = h(x_1, \dots, x_n)$ in $\mathcal{T}$ when co-interpreted in $\mathscr{A}$ holds. The *co-interpretation* of an n-ary expression $g(x_1, \dots, x_n)$ is given by the recursive rules.

0) The co-interpretation of $f_i \in \mathcal{T}$ is $\bar{f}_i : A \to \Sigma_{v_i} A$.

1) If $g(x_1, \dots, x_n) = x_j$, then $\bar{g} = u_j : A \to \Sigma_n A$, where u_j is the canonical injection appearing in the definition of Σ.

2) If $g(x_1, \dots, x_n) = f_i(h_1(x_1, \dots, x_n), h_2(x_1, \dots, x_n), \dots, h_{v_i}(x_1, \dots, x_n))$ and if we have already interpreted the h_i's as maps $\bar{h}_i : A \to \Sigma_n A$, then we interpret g as the composition

$$A \xrightarrow{\bar{f}_i} \Sigma_n A \xrightarrow{\Sigma \bar{h}_i} \Sigma_n A .$$

Let us specialize to a very special case. Let $\mathcal{T}$ be the theory for groups and $\mathscr{H}$ the category of spaces with base points and homotopy classes of maps. $\mathscr{H}$ is an abstract category. It is, if you will, a quotient of the concrete category of spaces and continuous maps, obtained via the homotopy congruence relation. The sum of two spaces A and B in $\mathscr{H}$ is constructed as the "wedge sum" $A \vee B$, by taking the disjoint union of A and B and then identifying their base points. If A is a $\mathcal{T}$-co-algebra, or as usually said in this context, a *co-group*, then its "co-multiplication" is a map $A \to A \vee A$. The best known co-groups are the spheres. If S^n is the n-sphere, note that if the equator $S^{n-1} \subset S^n$ is collapsed to a point, the result is homeomorphic to $S^n \vee S^n$. This collapsing is the co-multiplication $S^n \to S^n \vee S^n$. The constant map is the map which sends everything to the base point. The associativity equation does not hold in the category of continuous maps but it does hold in $\mathscr{H}$.

If A is $\mathcal{T}$-co-algebra in $\mathscr{A}$, then for any $B \in \mathscr{A}$, (A, B) acquires a $\mathcal{T}$-structure — not a co-structure. Hence the co-multiplication on spheres makes (S^n, B) into a group. And, of course, it is the usual multiplication on the usual homotopy groups.

The canonical co-structure on free algebras. Let $\mathcal{T}$ be an algebraic theory and let F be the free algebra generated by $x \in F$ in $\mathscr{S}^{\mathcal{T}}$. F has a canonical $\mathcal{T}$-co-structure in $\mathscr{S}^{\mathcal{T}}$. (Note that F need not have a $\mathcal{T}$-structure in $\mathscr{S}^{\mathcal{T}}$.) First observe that $\Sigma_n F$ is the free algebra on n generators $\{x_1, \dots, x_n\}$. The canonical co-interpretation of $f_i \in \mathcal{T}$ is defined to be the map $\bar{f}_i : F \to \Sigma_{v_i} F$ which sends x into the expression $f_i(x_1, \dots, x_{v_i}) \in \Sigma_{v_i} F$. If g is any n-ary expression in $\mathcal{T}$, then the co-interpretation of g turns out to be the map $g : F \to \Sigma_n F$ which sends x into $g(x_1, \dots, x_n) \in \Sigma_n F$ and the equations of $\mathcal{T}$ hold for the canonical co-interpretations.

The $\mathbb{T}$-co-structure on F makes the set of maps (F, B) into an algebra. Of course the set-valued functor $(F, -)$ is naturally equivalent to the forgetful functor $\mathscr{S}^{\mathbb{T}} \to \mathscr{S}$ (it forgets the structure). When we view $(F, -)$ as an algebra valued functor, it is naturally equivalent to the identity functor $\mathscr{S}^{\mathbb{T}} \to \mathscr{S}^{\mathbb{T}}$. The canonical co-structure we have described on F is characterized by this last fact: it is the only co-structure (up to isomorphism) which makes $(F, -)$ naturally equivalent to the identity functor.

Let $T : \mathscr{A} \to \mathscr{B}$ be a contravariant functor which carries finite sums into products. T will then carry any $\mathbb{T}$-co-algebra in $\mathscr{A}$ into a $\mathbb{T}$-algebra in $\mathscr{B}$. If $\mathscr{A}$ is an algebraic category, then $T(F)$, for F the free algebra in $\mathscr{A}$, carries a natural $\mathbb{T}$-structure in $\mathscr{B}$. In other words, $T(F) \in \mathscr{B}^{\mathbb{T}}$. Referring back to theorem 1, if T is a functor satisfying the two conditions of that theorem, then perforce it carries finite sums into finite products and $T(F) \in \mathscr{S}^{\mathbb{T}_1 \otimes \mathbb{T}_2}$.

Now if T is representable, then $T(F)$ is isomorphic to (F, A) as a $\mathbb{T}_1$-algebra, and (F, A) is isomorphic to A. Hence if we wish to find the representor of T we need only evaluate T on the free algebra $F \in \mathscr{S}^{\mathbb{T}_1}$ and obtain the $\mathbb{T}_1 \otimes \mathbb{T}_2$-algebra $T(F) \in \mathscr{S}^{\mathbb{T}_1 \otimes \mathbb{T}_2}$. In the construction of injective modules we start with the free algebra in the category of right R modules, namely, R itself. R is simultaneously a left-R module (there's the co-structure). We consider the set of maps $(R, Q/Z)$, where R is momentarily considered to be just an abelian group. But the left R-module structure on R makes $(R, Q/Z)$ into a right R-module. Such is an injective cogenerator. (This fact was first observed by Eckmann and Schopf [3]).

Theories for autonomous categories. If $\mathbb{T}$ is the theory of groups, then the only co-groups in $\mathscr{S}^{\mathbb{T}}$ are the free groups. If $\mathbb{T}$ is the theory of *abelian* groups, then for every $A \in \mathscr{S}^{\mathbb{T}}$ there is a unique $\mathbb{T}$-co-structure on A, namely that for which the co-multiplication $\overline{m} : A \to A \oplus A$ is such that $\overline{m}(x) = \langle x, x \rangle$ $(A + A \simeq A \times A$ in $\mathscr{S}^{\mathbb{T}})$. For $A, B \in \mathscr{S}^{\mathbb{T}}$, (A, B) acquires a group structure by either the group structure on B or the co-group structure on A; they are the same. Note that B has a unique $\mathbb{T}$-structure in $\mathscr{S}^{\mathbb{T}}$. That is $(\mathscr{S}^{\mathbb{T}})^{\mathbb{T}} = \mathscr{S}^{\mathbb{T}}$, $\mathbb{T} \otimes \mathbb{T} = \mathbb{T}$.

In general, if $\mathbb{T}$ is a theory such that there exists a functor $\mathrm{Hom}(A, B)$ with values in $\mathscr{S}^{\mathbb{T}}$ for $A, B \in \mathscr{S}^{\mathbb{T}}$ such that the underlying set of $\mathrm{Hom}(A, B)$ is the set of maps from A to B, then each $B \in \mathscr{S}^{\mathbb{T}}$ has a canonical $\mathbb{T}$-structure in $\mathscr{S}^{\mathbb{T}}$ (we have an embedding $\mathscr{S}^{\mathbb{T}} \to \mathscr{S}^{\mathbb{T} \otimes \mathbb{T}}$), and each $A \in \mathscr{S}^{\mathbb{T}}$ has a canonical $\mathbb{T}$-co-structure in $\mathscr{S}^{\mathbb{T}}$. We can conclude that $\mathbb{T}$ must be such that each $\mathbb{T}$-operator is a $\mathbb{T}$-homomorphism. As in my examination of $\mathbb{T}_1 \otimes \mathbb{T}_2$, we can prove that such a $\mathbb{T}$ has at most one constant, and that if it has a binary operation with zero, then $\mathbb{T}$ is the theory of modules over a *commutative* half-ring.

Linton, in his Columbia dissertation [8] calls a category *autonomous* if it has a *forgetful* functor $F: \mathscr{A} \to \mathscr{S}$ and a functor $\mathrm{Hom}: \mathscr{A}^* \times \mathscr{A} \to \mathscr{S}$ such that $F(\mathrm{Hom}(A, B)) \simeq (A, B)$. Hence we have just identified those theories $\mathbb{T}$ such that $\mathscr{S}^{\mathbb{T}}$ is autonomous.

Let $\mathscr{A}$ be an arbitrary category with finite products. What is the largest algebraic theory $\mathbb{T}$ such that we can factor $\mathrm{Hom}: \mathscr{A}^* \times \mathscr{A} \to \mathscr{S}$ through the forgetful functor $\mathscr{S}^{\mathbb{T}} \to \mathscr{S}$? $\mathbb{T}$ may be constructed as the algebraic theory of the identity functor $I \in (\mathscr{A}, \mathscr{A})$.

$\mathbb{T}$ is necessarily such that $\mathscr{S}^{\mathbb{T}}$ is autonomous. If $\mathscr{A}$ is the category of left R-modules, $\mathbb{T}$ will be the theory of modules over the center of R. In general, if $\mathscr{A} = \mathscr{S}^{\mathbb{T}}$, then $\mathbb{T}$ is the subtheory of $\mathbb{T}'$ generated by those expressions in $\mathbb{T}'$ which are $\mathbb{T}'$ homomorphism.

Co-constant maps and 0'ary co-operations. If f is a unary operation in $\mathbb{T}$, then the co-interpretation of the equation $f(x) = f(y)$ is

$$A \xrightarrow{\bar{f}} A \xrightarrow{u_1} A + A = A \xrightarrow{f} A \xrightarrow{u_2} A + A$$

which is equivalent to $A \xrightarrow{\bar{f}} A \xrightarrow{g} B = A \xrightarrow{\bar{f}} A \xrightarrow{h} B$ for all $g, h \in (A, B)$, all B.

We shall call such a map a *co-constant operation.*

If f is a 0'ary operation in $\mathbb{T}$, then its co-interpretation is a choice $\{f_B \in (A, B)\}_{B \in \mathscr{A}}$ such that $A \xrightarrow{f_B} B \to B = A \xrightarrow{f_{B'}} B'$ for all $B \to B' \in \mathscr{A}$.

If $(A, B) \neq \emptyset$ all $B \in \mathscr{A}$, then the co-constant operations on A and the 0'ary operations on A are in one-to-one correspondence.

If $(A, B) = \emptyset$ some $B \in \mathscr{A}$, then there are no 0'ary operations on A.

Now for the free-algebra $F \in \mathscr{S}^{\mathbb{T}}$, (F, B) is empty only if B is empty, hence only if $\mathbb{T}$ has no 0'ary operations. Otherwise, if $\mathbb{T}$ does have 0'ary operations, then the 0'ary co-operations on F correspond to the coconstant operations and they in turn correspond to the 0'ary operations in $\mathbb{T}$.

In general, all co-constant operations on $A \in \mathscr{A}$ must be maps all of whose values are algebraic constants. And conversely, any $A \to A$ the image of which is in the atomic subalgebra of A is a co-constant operation.

Covariant representable functors. We shall say that a covariant functor $T: \mathscr{A} \to \mathscr{S}^{\mathbb{T}}$ is *representable* if there exists a $\mathbb{T}$-co-algebra $A \in \mathscr{A}$ such that T is naturally equivalent to the functor $(A, -)$. If $\mathbb{T}$ is the empty theory, this definition coincides with the standard: $\mathscr{S}^{\mathbb{T}} = \mathscr{S}$ and $(A, -)$ is the usual set valued functor.

I shall momentarily use the language of [4] to state and sketch the proof of a theorem which will be later translated back into the language of general algebra.

THEOREM 2. *Let $\mathscr{A}$ be a complete category, $\mathbb{T}$ an equational algebraic theory, and $T : \mathscr{A} \to \mathscr{S}^{\mathbb{T}}$ a covariant functor. T is representable if and only if T has a left-adjoint.*

Proof. I shall not here construct the left-adjoint of a representable functor. The latter sections on tensor products of algebras strongly suggest the construction for general $\mathscr{A}$, and indeed, contains the construction for the case that $\mathscr{A}$ is an algebraic category.

Suppose that $T : \mathscr{A} \to \mathscr{S}^{\mathbb{T}}$ does have a left-adjoint $S : \mathscr{S}^{\mathbb{T}} \to \mathscr{A}$. Let F be the free algebra on one generator in $\mathscr{S}^{\mathbb{T}}$. The adjointness of T and S says that the set-valued functors $(F, T(-))$ and $(S(F), -)$ are naturally equivalent. The canonical $\mathbb{T}$-co-structure on F induces a $\mathbb{T}$-co-structure on $S(F)$ (left adjoints always preserve sums), and if we view $(F, T(-))$ and $(S(F), -)$ as $\mathbb{T}$-algebra valued functors, the natural equivalence is still an equivalence (it is a homomorphism of algebras because of its naturality on the first variable). Of course, as we have already remarked $(F, -)$ is naturally equivalent to the identity functor, hence $T(-)$ is naturally equivalent to $(S(F), -)$, q. e. d.

COROLLARY. *Let $\mathscr{A}$ be a complete category, $\mathbb{T}$ an equational algebraic theory, and $T : \mathscr{A} \to \mathscr{S}^{\mathbb{T}}_{*}$ a contravariant functor. T has an adjoint on the right if and only if T is representable.*

Proof. Just replace $\mathscr{A}$ with $\mathscr{A}^*$.

For the proof of Theorem 1 use the Special Adjoint Functor Theorem of [4] together with the observations that first, an algebraic category always has a generator (the free algebra) and second, the two conditions of theorem 1 are equivalent to the statement that T carries right-roots into left-roots, q. e. d.

It will be a little more difficult to translate the covariant case into general algebra. The special adjoint functor theorem can not be used because an algebraic category need not have a co-generator (e. g. the category of groups, abelian or not). We shall need the general adjoint functor theorem.

The continuity conditions are easy to translate and will be stated in the theorem below. The solution set condition needs analysis. It turns out that it is easier to translate the conditions if we stipulate the size of the representor. Let $A \in \mathscr{S}^{\mathbb{T}_1}$ be a finitely generated $\mathbb{T}_1$-algebra with a $\mathbb{T}_2$-co-structure. Let $\{B_i\}$ be a *directed* family of subalgebras of $B \in \mathscr{S}^{\mathbb{T}_1}$, that is, every finite subfamily in $\{B_i\}$ is bounded in $\{B_i\}$. Note that if we consider (A, B_i) as a subset of (A, B), then

$$\bigcup (A, B_i) = (A, \bigcup B_i).$$

This property of $(A, -)$, namely that it preserves directed unions is, in fact, equivalent with the property that A be finitely generated.

THEOREM 3. *A functor* $T: \mathscr{S}^{\mathbb{T}_1} \to \mathscr{S}^{\mathbb{T}_2}$ *is representable by a finitely generated* $\mathbb{T}_1$*-algebra (with a* $\mathbb{T}_2$*-co-structure) if and only if*

1) T *preserves products,*

2) T *preserves difference kernels: if* $f_1, f_2 \in (B, C) \subset \mathscr{S}^{\mathbb{T}_1}$ *and* $K = \{x \in B \mid f_1(x) = f_2(x)\}$, *then* $T(K \to B)$ *is a one-to-one map and its image is* $\{x \in T(B) \mid (Tf_1)(x) = (Tf_2)(x)\}$.

3) T *preserves directed unions: if* $\{B_i\}$ *is a directed family of subalgebras of* B, *then* $\bigcup T(B_i) = T(\bigcup B_i)$.

Proof. In the language of [4] we wish to show that if $B \in \mathscr{S}^{\mathbb{T}_1}$ is generated through T by $F \in \mathscr{S}^{\mathbb{T}_2}$, then B is finitely generated. Officially we wish to show more according to 3,L in [4], but this is a typical case.

Consider the generating map $F \to T(B)$. Let $\{B_i\}$ be the directed family of finitely generated subalgebras of B.

$\bigcup B_i = B$ and hence the map $F \to T(B)$ must factor through some $T(B_i)$. By the definition of the phrase "generated by F through T" we conclude that for some $i, B_i = B$, and that B is finitely generated, q. e. d.

The general result for covariant representable functors between algebraic categories needs the following definition, where L is any infinite cardinal number.

An *L-directed family* of subalgebras is a family such that every subfamily of cardinality L or less is bounded within the family.

THEOREM 4. *A functor* $T: \mathscr{S}^{\mathbb{T}_1} \to \mathscr{S}^{\mathbb{T}_2}$ *is representable by a* $\mathbb{T}_1$*-algebra with* L *or less generators if* 1) T *preserves products,* 2) *difference kernels, and* 3) L*-directed unions.*

Lawvere functors. A *Lawvere functor* is a functor $T: \mathscr{S}^{\mathbb{T}_1} \to \mathscr{S}^{\mathbb{T}_2}$ which preserves underlying sets. The easiest are those that "forget" some of the structure, e. g., the functor that sends a ring to its multiplicative semi-group, or the functor that sends an R-module to its underlying additive group. An example of a Lawvere functor which is not forgetful is that which sends an associative algebra to a Lie-algebra by defining $[x, y] = xy - yx$.

A Lawvere functor trivially preserves products, difference kernels and directed unions and hence by Theorem 3 it is representable. Suppose $A \in \mathscr{S}^{\mathbb{T}_1}$ represents a Lawvere functor. Because there is a natural equivalence between the *sets* B and (A, B) it must be the case that A is the free algebra on one generator in $\mathscr{S}^{\mathbb{T}_1}$. Hence the Lawvere functors from $\mathscr{S}^{\mathbb{T}_1}$ to $\mathscr{S}^{\mathbb{T}_2}$ are in one-to-one correspondence with the $\mathbb{T}_2$-co-structures that may be placed on the $\mathbb{T}_1$ free algebra. And these in turn are in one-to-one correspondence with the *theory-maps* $\mathbb{T}_2 \to \mathbb{T}_1$, as described below.

Consider a $\mathbb{T}_2$-co-structure on the free algebra $F \in \mathscr{S}^{\mathbb{T}_1}$. For each $f_i \in \mathbb{T}_2$ there is assigned $\bar{f}_i : F \to \Sigma_{v_i} F$. Bur f_i is determined by $f_i(x)$, for x the generator of F, and $f_i(x)$ is just an n-ary expression in $\mathbb{T}_1$. For each operation in $\mathbb{T}_2$ we will obtain a $\mathbb{T}_2$-expression of the same valence, and for each equation in $\mathbb{T}_2$ we will obtain a $\mathbb{T}_1$-equation if we replace the operators with the corresponding expressions. Such is what we mean by a *theory-map* $\mathbb{T}_2 \to \mathbb{T}_1$. Conversely any theory-map yields a $\mathbb{T}_2$-co-structure on $F \in \mathscr{S}^{\mathbb{T}_1}$.

A familiar example is the case when we are given a ring homomorphism $R_2 \to R_1$. Such may be interpreted as a map between the algebraic theories of their modules. The corresponding Lawvere functor is the familiar change-of-rings functor.

When $\mathbb{T}_2$ is a sub-theory of $\mathbb{T}_1$, perhaps fewer operators, perhaps fewer equations, then the Lawvere functor $\mathscr{S}^{\mathbb{T}_1} \to \mathscr{S}^{\mathbb{T}_2}$ is the forgetful functor.

When $\mathbb{T}_2$ is the theory of Lie-algebras, $\mathbb{T}_1$ the theory of associative algebras, and $\mathbb{T}_2 \to \mathbb{T}_1$ the theory map that is constant on $0, +, -$ and sends $[x, y]$ into $xy - yx$, then the Lawvere functor $\mathscr{S}^{\mathbb{T}_1} \to \mathscr{S}^{\mathbb{T}_2}$ is the previously mentioned example.

Remark. Abandoning formal linguistic inhibitions we obtain a contravariant functor from the category of algebraic theories and theory maps to the category of categories. Lawvere in [8] [6] calls this the Semantics Functor. He identifies therein its adjoint which he calls the Algebraic Structure Functor. Semantics is adjoint to Structure.

Tensor products of algebras. Let $\mathbb{T}_1$, $\mathbb{T}_2$ be algebraic theories. We shall denote the category of $\mathbb{T}_1$-co-algebras in $\mathscr{S}^{\mathbb{T}_2}$ by the notation ${}_{\mathbb{T}_1}\mathscr{S}^{\mathbb{T}_2}$. Given $A \in {}_{\mathbb{T}_2}\mathscr{S}^{\mathbb{T}_1}$, $B \in {}_{\mathbb{T}_3}\mathscr{S}^{\mathbb{T}_2}$ consider the composition

$$\mathscr{S}^{\mathbb{T}_1} \xrightarrow{(A,-)} \mathscr{S}^{\mathbb{T}_2} \xrightarrow{(B,-)} \mathscr{S}^{\mathbb{T}_3}.$$

By theorem 4 the composition is representable by a $\mathbb{T}_3$-co-algebra in $\mathscr{S}^{\mathbb{T}_1}$. Let us call it $B \otimes A \in {}_{\mathbb{T}_3}\mathscr{S}^{\mathbb{T}_1}$:

$$\text{(t)} \qquad (B \otimes A, C) \simeq \big(B, (A, C)\big).$$

A little general functor theory [4] makes $\otimes$ into a two variable functor

$${}_{\mathbb{T}_3}\mathscr{S}^{\mathbb{T}_2} \otimes {}_{\mathbb{T}_2}\mathscr{S}^{\mathbb{T}_1} \to {}_{\mathbb{T}_3}\mathscr{S}^{\mathbb{T}_1}$$

and the isomorphisms (t) are natural in all three variables. In particular $- \otimes A : \mathscr{S}^{\mathbb{T}_2} \to \mathscr{S}^{\mathbb{T}_1}$ is the adjoint of $(A, -) : \mathscr{S}^{\mathbb{T}_1} \to \mathscr{S}^{\mathbb{T}_2}$ which is just a fancy way of saying that equation (t) holds and is natural in B and C.

A few examples: First are the classical usages of tensor product. The tensor product of rings is *not* an example of my tensor product of algebras, it is an example of the tensor product of theories. Tensor products of modules, however, are examples: Let T_2 be the theory of right R_2-modules ,T_1 the theory of abelian groups. An object $A \in {}_{T_2}\mathscr{S}^{T_1}$ is just a *left* R-module: $B \in \mathscr{S}^{T_2}$ is a right R-module. $B \otimes A \in \mathscr{S}^{T_1}$ is a group. In this case we have taken T_3 to be empty.

Let T_1 be the theory of right-R_1-modules, T_3 the theory of right R_3-modules. Then $A \in {}_{T_2}\mathscr{S}^{T_1}$ is a group with a right R_1 module structure and a left R_2 module such that $r_2(xr_1) = (r_2x)r_1$ for all $\langle r_1, x, r_2\rangle \in R_1 \times A \times R_2$. Similarly $B \in {}_{T_3}\mathscr{S}^{T_2}$ is a right R_2-module, left R_3-module and $r_3(xr_2) = (r_3x)r_2$. $B \otimes A \in {}_{T_3}\mathscr{S}^{T_1}$ is right R_1-module and a left R_3-module.

The isomorphism (t) gives the universal mapping property of $B \otimes A$. A map $B \to (A, C)$ is an R_2-bilinear map.

Let T_1 be the theory of rings with identity, T_2 the theory of groups, $T : \mathscr{S}^{T_1} \to \mathscr{S}^{T_2}$ the functor which sends a ring to its group of units. T may easily be seen to verify the condition of theorem 3 and hence is representable, say be $A \in {}_{T_2}\mathscr{S}^{T_1}$. The isomorphism (t) identifies $B \otimes A \in \mathscr{S}^{T_1}$, for $B \in \mathscr{S}^{T_2}$, as the group ring of B. In this fashion we can identify $A : Z \otimes A$ is the group ring of Z. But $(Z \otimes A, C) \simeq \big(Z, (A, C)\big) \simeq (A, C)$ for all C and hence $Z \otimes A = A$. If we construct A as $Z[X, Y]/(XY-1, YX-1)$ or more conveniently as $Z[X, X^{-1}]$ the Z polynomials with positive and negative coefficients, then we can construct its T_1-co-structure as that given by the co-multiplication

$$\overline{m} : Z[X, X^{-1}] \to Z[X, X^{-1}] + Z[X, X^{-1}] \simeq Z[X_1, X_2, X_1^{-1}, X_2^{-1}-1]$$

where $\overline{m}(X) = (X_1 X_2)$.

PROPOSITION. *Tensor products of algebras are associative. That is, $C \otimes (B \otimes A)$ is naturally equivalent to $(C \otimes B) \otimes A$.*

Proof. Immediately from the definition of $B \otimes A$ as that which represents $\big(B, (A, -)\big)$.

Characterization of functors given by tensoring. We now prove

THEOREM 5. *A covariant functor $T : \mathscr{S}^{T_1} \to \mathscr{S}^{T_2}$ is naturally equivalent to $-\otimes A : \mathscr{S}^{T_1} \to \mathscr{S}^{T_2}$ where $A \in {}_{T_1}\mathscr{S}^{T_2}$ if and only if*

1) *T preserves free sums,*

2) *T preserves difference co-kernels: If $f : B \to B''$ is an onto map in $\mathscr{S}^{T_1}$, $K = \{\langle x_1, x_2\rangle \in B \times B \mid f(x_1) = f(x_2)\}$ and $p_i(x_1, x_2) = x_i$, then $T(B) \to T(B'')$ is onto and the congruence it defines on TB is generated by pairs $\{\langle y_1, y_2\rangle \in T(B) \times T(B) \mid$ there is $z \in T(K)$, $T(p_i)(z) = y_i\}$.*

Proof. That tensor products enjoy the two properties is a formal consequence of the adjointness relation (t), as described in [4]. For the other direction we may rely on the Special Adjoint Functor Theorem, q. e. d.

If we describe a construction, we would note that A as a $\mathbb{T}_2$-algebra is $T(F)$, where F is the free algebra in $\mathscr{S}^{\mathbb{T}_1}$. A acquires a $\mathbb{T}_1$-co-structure from the canonical co-structure on F and the fact that T preserves sums.

The general functor approach indicates that we should be able to define a more general tensor product $\mathscr{S}^{\mathbb{T}} \otimes {}_{\mathbb{T}}\mathscr{A}$ where $\mathscr{A}$ is an arbitrary right complete category and ${}_{\mathbb{T}}\mathscr{A}$ is the category of $\mathbb{T}$-co-algebras in $\mathscr{A}$. And indeed we can. Or could if so inclined.

Moreover we could define a "symbolic hom" functor $(\mathscr{S}^{\mathbb{T}}, \mathscr{A}^{\mathbb{T}})$ contravariant on the first variable, covariant on the second. And the isomorphism (t) would still hold.

Generators and relations for tensor products of algebras. The formal properties of $\otimes$ as described in Theorem 5 tell us that there is a scheme for generators and relations. If we represent $B \in {}_{\mathbb{T}_3}\mathscr{S}^{\mathbb{T}_1}$ as a free algebra $\Sigma_n F$ modulo a congruence K, then the difference co-kernel preservation will yield a congruence on $\Sigma_n F \otimes A = \Sigma_n A$ which defines $B \otimes A$. As for the $\mathbb{T}_3$-co-structure on B, it transfers to $B \otimes A$ simply by the sum-preservation property of $-\otimes A$.

We shall, however, directly argue the proof of the following theorem:

THEOREM 6. *Let $B \in \mathscr{S}^{\mathbb{T}_1}$, $A \in {}_{\mathbb{T}_1}\mathscr{S}^{\mathbb{T}_2}$. $B \otimes A$ is the $\mathbb{T}_2$-algebra generated by $\{b \otimes a \mid b \in B, a \in A\}$ subject to the relations*

Type I: for each $f_i \in \mathbb{T}_2$, $b \in B$, $a_1, \ldots, a_{v_i} \in A$

$$b \otimes f_i(a_1, \ldots, a_{v_i}) = f_i(b \otimes a_1, \ldots, b \otimes a_{v_i});$$

Type II: for each $g_i \in \mathbb{T}_1$, $b_1, \ldots, b_{v_i} \in B$, $a \in A$ let $a_1, \ldots, a_{v_i} \in A$ be such that the co-operator $\bar{g}_i : A \to A$ is such that $g_i(a) = h(a_1, \ldots, a_{v_i})$ where h is a v_i-expression in $\mathbb{T}_2$:

$$g_i(b_1, \ldots, b_{v_i}) \otimes a = h(b_1 \otimes a_1, \ldots, b_{v_i} \otimes a_{v_i}).$$

Proof. For the purposes of the proof I shall suppose that $B \otimes A$ is *defined* by the generators and relations in the theorem, and I shall construct the isomorphisms $\varphi : (B \otimes A, C) \to (B, (A, C))$ and $\psi : (B, (A, C)) \to (B \otimes A, C)$.

Definition of φ. Given a $\mathbb{T}_2$-map $F : B \otimes A \to C$, φF must be a $\mathbb{T}_1$-map from B to (A, C).

Given $b \in B$, $\varphi F(b)$ must be a $\mathbb{T}_2$-map from A to C.

Given $a \in A$, $(\varphi F(b))(a)$ must be an element in C.

Define $(\varphi F(b))(a) = F(b \otimes a)$.

First verification. $\varphi F(b)$ is a $\mathbb{T}_2$-map. Use Type I relations.

Second verification. φF is a $\mathbb{T}_1$-map. Use Type II relations.

Definition of ψ. Given a $\mathbb{T}_1$-map $G: B \to (A, C)$, ψG must be a $\mathbb{T}_2$-map from $B \otimes A$ to C.

Given $b \in B$, $a \in A$, $\psi G(b \otimes a)$ must be an element in C.

Define $\psi G(b \otimes a) = (G(b))(a)$ and extend to a homomorphism.

First verification. ψG can indeed be extended to all of $B \otimes A$. One must show that the type I and II equations do not obstruct.

Second verification. ψG is a $\mathbb{T}_2$-map. Automatic because ψG was defined on the generators and extended.

Finally: φ and ψ are inverses of each other.

$\varphi\psi G = G$ because

$$(\varphi\psi G)(b)(a) = \psi G(b \otimes a) = (G(b))(a)$$

and $\psi\varphi F = F$ because

$$(\psi\varphi F)(b \otimes a) = (\varphi F(b))(a) = F(b \otimes a).$$

COROLLARY. *If B is a finitely generated $\mathbb{T}_1$-algebra and A is finitely generated as a $\mathbb{T}_2$-algebra, then so is $B \otimes A$.*

If further, B and A are finitely related, then so is $B \otimes A$.

Co-Lawvere functors. Given a theory-map $\mathbb{T}_2 \to \mathbb{T}_1$, we have seen that the corresponding Lawvere functor $\mathscr{S}^{\mathbb{T}_1} \to \mathscr{S}^{\mathbb{T}_2}$ is represented by the free algebra $F \in \mathscr{S}^{\mathbb{T}_1}$ where F has a $\mathbb{T}_2$-co-structure induced by the theory-map $\mathbb{T}_2 \to \mathbb{T}_1$. The *Co-Lawvere functor* $-\otimes F: \mathscr{S}^{\mathbb{T}_2} \to \mathscr{S}^{\mathbb{T}_1}$ has an even simpler construction then for general tensor products. Because F is generated by a single element one can show that $B \otimes F$ is the $\mathbb{T}_1$-algebra generated by $\{\bar{b} \mid b \in B\}$ subject to the relations,

$$\overline{f_i(b_1, \ldots, b_{v_i})} = g_i(\bar{b}_1, \ldots, \bar{b}_{v_i}) \quad \text{where} \quad f_i \in \mathbb{T}_2$$

and g_i is the $\mathbb{T}_1$-expression determined by the theory map $\mathbb{T}_2 \to \mathbb{T}_1$.

If all the operators in $\mathbb{T}_1$ have ancestors in $\mathbb{T}_2$, e. g. when $\mathbb{T}_1$ can be obtained from $\mathbb{T}_2$ by throwing away some equations, then the Lawvere functor $\mathscr{S}^{\mathbb{T}_1} \to \mathscr{S}^{\mathbb{T}_2}$ is just an inclusion and the co-Lawvere functor $\mathscr{S}^{\mathbb{T}_2} \to \mathscr{S}^{\mathbb{T}_1}$ sends $B \in \mathscr{S}^{\mathbb{T}_1}$ to B mod the congruence generated by the additional equations.

If $\mathbb{T}_2 \to \mathbb{T}_1$ is one-to-one, i. e. $\mathbb{T}_2$ can be obtained from $\mathbb{T}_1$ by throwing away some operators, then the Lawvere functor $\mathscr{S}^{\mathbb{T}_1} \to \mathscr{S}^{\mathbb{T}_2}$ is simply a forgetful functor and the co-Lawvere functor is an inflationary operation.

Automorphisms on algebraic categories. Let $T\colon \mathscr{S}^{\mathbb{T}} \to \mathscr{S}^{\mathbb{T}}$ and $S\colon \mathscr{S}^{\mathbb{T}} \to \mathscr{S}^{\mathbb{T}}$ be functors such that TS and ST are naturally equivalent to the identity functors. Because the hypotheses and conclusions of the condition of Theorem 3 are all equivalent to categorically definable statements and because the conditions are true for the identity functor, they are true for T and S. Hence T and S are representable by finitely generated algebras:

$$T \simeq (A, -), \qquad S \simeq (B, -), \qquad A, B \in_{\mathbb{T}} \mathscr{S}^{\mathbb{T}}.$$

It follows that $A \otimes B$ and $B \otimes A$ are each isomorphic, as objects in ${}_{\mathbb{T}}\mathscr{S}^{\mathbb{T}}$, to the free algebra with its canonical co-structure, and the (t) isomorphism says that $T \simeq - \otimes B$, $S \simeq - \otimes A$.

If we consider the isomorphism classes of finitely generated objects in ${}_{\mathbb{T}}\mathscr{S}^{\mathbb{T}}$ we obtain a semi-group under $\otimes$ with neutral element F. The group of units is isomorphic to the automorphism-class-group of $\mathscr{S}^{\mathbb{T}}$, defined in [4] as the group of natural equivalence classes of automorphisms. We shall denote that group by $\mathbb{A}(\mathscr{S}^{\mathbb{T}})$.

Let $\mathbb{A}(\mathbb{T})$ be the group of automorphisms of theory-maps of $\mathbb{T}$. Let $\mathscr{V}(\mathbb{T})$ be the unary expressions of $\mathbb{T}$ considered as a semi-group and let $G(\mathbb{T})$ be the group of units in $\mathscr{V}(\mathbb{T})$. For $f \in G(\mathbb{T})$ the map $\varphi : \mathbb{T} \to \mathbb{T}$ defined by

$$\varphi\big(g(x_1, \ldots, x_n)\big) = f^{-1}g\big(f(x_1), \ldots, f(x_n)\big)$$

is a theory map. We obtain map $G(\mathbb{T}) \to \mathbb{A}(\mathbb{T})$, and we call its image $\mathbb{I}\mathbb{A}(\mathbb{T})$, the group of *inner-automorphisms* on $\mathbb{T}$.

PROPOSITION. *$\varphi \in \mathbb{I}\mathbb{A}(\mathbb{T})$ if and only if the φ-co-structure induced on F is isomorphic to the canonical co-structure.*

Define $\mathbb{S}\mathbb{A}(\mathscr{S}^{\mathbb{T}})$ to be the automorphisms that leave the free algebra unchanged (up to isomorphisms).

PROPOSITION. $\mathbb{S}\mathbb{A}(\mathscr{S}^{\mathbb{T}}) \simeq \mathbb{A}(\mathbb{T})/\mathbb{I}\mathbb{A}(\mathbb{T})$.

If $\mathscr{S}^{\mathbb{T}}$ is an autonomous category, then $\mathbb{S}\mathbb{A}(\mathscr{S}^{\mathbb{T}})$ is a normal subgroup of $\mathbb{A}(\mathscr{S}^{\mathbb{T}})$. In general it is not. The index of $\mathbb{S}\mathbb{A}$ in $\mathbb{A}$ is as big as the orbit of F, and it measures our inability to characterize the free algebra using only category predicates.

Remark (*added in proof*). Lawvere has suggested that tensor products of theories be called Kronecker products. Though the same symbol $\otimes$ be retained, such usage will indeed remove a source of confusion (see p. 93).

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