An introduction to statistical modelling semantics with higher-order measure theory

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Spaces Statistical Modelling needs:

Discrete finite

Bool

true false

Animal

cat
dog
Giraffe

Discrete Countable

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, String

Continuous

$\mathbb{R}^n$ $\mathbb{R}^{\mathbb{N}}$

$\mathcal{P}\mathbb{R}$, Weight

standard Borel spaces

Measurable

Recent developments:

Discrete finite

Discrete Countable

Continuous

Regular ordered Banach

[Dahlqvist-Kozen '20]

Probabilistic Coherence Spaces & Measurable Cones

[Ehrhard-Pagani-Tasson '18]

Boolean-Valued Models

[Bacci-et al '18]

interval domains

[Jia et al. '21]

~~Measurable~~

This talk

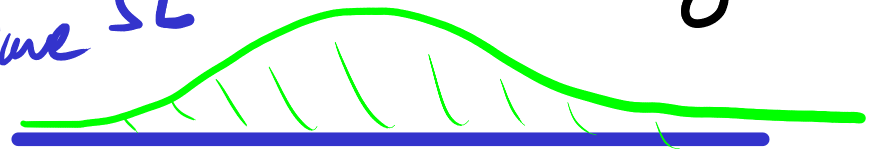
Quasi-Borel Spaces

[Heunen et al. 17]

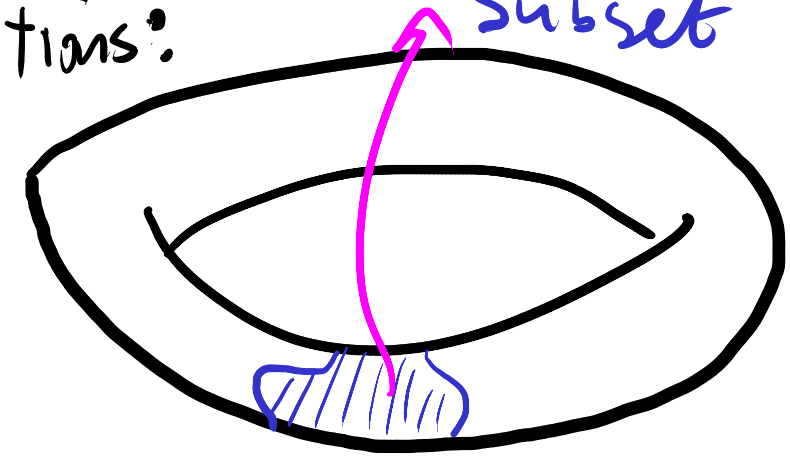
Core idea

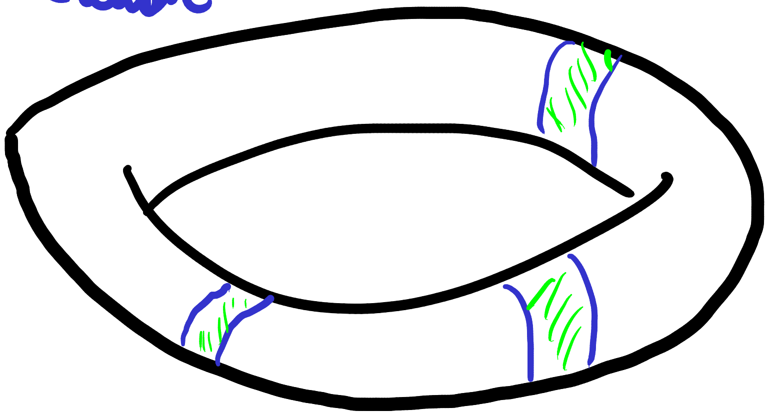
Measure Theory | **Qbs Theory**

sample space Ω



Primitive notions: measurable subset | random element $\downarrow I\alpha$





Derived notions: random elements $\alpha : \Omega \to \text{Space}$ | measure | measurable subsets

Conservative extensions:

concrete spaces we "observe"

Standard Borel spaces

Meas

Qbs

abstract spaces

(auxiliary?)

Wide topic:

Variations

Qbs, wQSs, [Vákár et al '19]

QMS, QUS, [Forré '21]

(w)Diff, wPap [Lew et al. '22]

[Vákár et al. 20-21]

Applications

MC inference

design & verification [Ścibior et al '17a+b]

Network Programming [Vandenbroucke-Schrijvers '19]

Semantics

name generation [Sabok et al. '21]

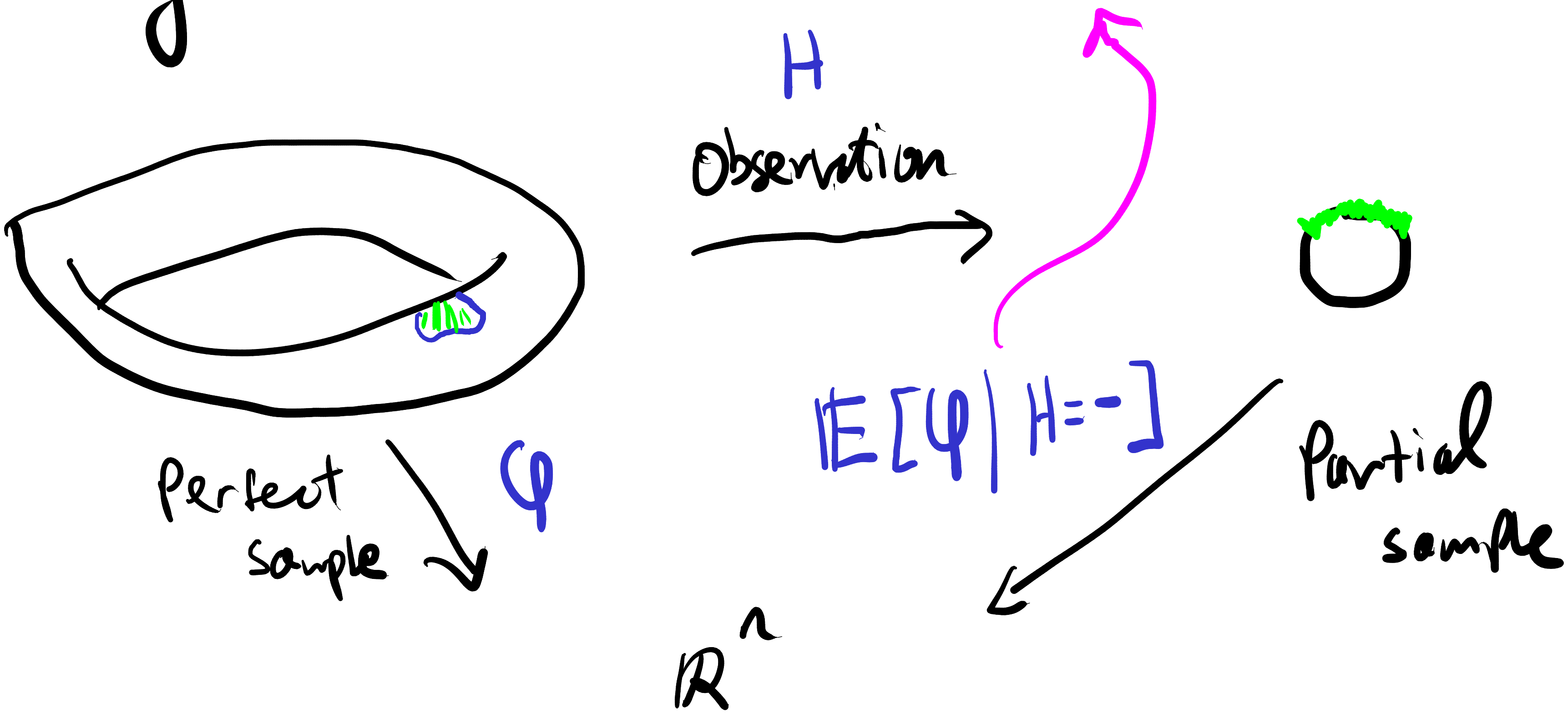
This course:

- Peek behind scenes
- Gain working knowledge

Theme: higher-order measure theory

demonstrated through

Kolmogorov's Conditional Expectation



Kolmogorov's Conditional Expectation

- naturally higher order: $\mathbb{R}^{\Omega} \to \mathbb{R}^{\Theta}$
- behind many modern Probability techniques:
  - existence of Radon-Nikodym derivatives & density
  - existence of disintegration
  - foundation of martingales & stochastic differential equations

Agenda

I
- ∘ Borel sets
- ∘ Qbs:
  def., constructions, partiality, type structure

II
- • Measures & integration
- • Random variable spaces
- • Conditional expectation

Slogan:

Measurable by Type

NB:

- Exercise Sheets 
- #qbs on SPLS Zulip

Space: all possible states

Subset: all states of current interest

Measure: probability/weight/length assignment to

fine for discrete spaces

e.g.

$\{H, T\}^5$

HHTTH

$\frac{1}{32}$

Continuous Caveat:

Thm: no $\lambda : \mathcal{P}\mathbb{R} \to [0, \infty]$:

$\lambda(a, b) = b - a$ (generalises length)

$\lambda(r + A) = \lambda A$ (translation invariant)

$\lambda\left(\biguplus_{n=0}^{\infty} A_n\right) = \sum_{n=0}^{\infty} \lambda A_n$ σ-additive

Workaround: only measure well-behaved subsets

Def: The Borel subsets $\mathcal{B}_{\mathbb{R}} \subseteq \mathbb{R}$:

- open intervals $(a,b) \in \mathcal{B}_{\mathbb{R}}$

Closure under σ-algebra operations:

$$\frac{}{\emptyset \in \mathcal{B}_{\mathbb{R}}} \qquad \frac{A \in \mathcal{B}_{\mathbb{R}}}{A^c := \mathbb{R} \setminus A \in \mathcal{B}} \qquad \frac{\vec{A} \in \mathcal{B}_{\mathbb{R}}^{\mathbb{N}}}{\bigcup_{n=0}^{\infty} A_n \in \mathcal{B}_{\mathbb{R}}}$$

empty set

complements

countable unions

Examples

discrete Countable: $\{r\} = \bigcap_{\varepsilon \in \mathbb{Q}^+} (r-\varepsilon, r+\varepsilon) \in B_{\mathbb{R}}$

I countable $\implies I = \bigcup_{r \in I} \{r\} \in B_{\mathbb{R}}$

closed intervals: $[a,b] = (a,b) \cup \{a,b\}$

Non-examples?

More complicated: analytic, Lebesgue

Def: Measurable space $V = (|V|, B_V)$

Set (carrier)

family of subsets $B_V \subseteq P|V|$

closed under σ-algebra operations:

$$\frac{}{\emptyset \in B_V} \qquad \frac{A \in B_V}{A^c := |V| \setminus A \in B_V} \qquad \frac{\vec{A} \in B_V^{\mathbb{N}}}{\bigcup_{n=0}^{\infty} A_n \in B_V}$$

empty set — complements — countable unions

Idea: Structure all spaces after the worst-case scenario

Examples

- Discrete spaces $\ulcorner X^{\text{meas}} \urcorner = (X, PX)$
- Euclidean spaces $\mathbb{R}^n$ — replace intervals with chests $\prod_{i=1}^{n} (a_i, b_i)$

  $\mathbb{R}^{\mathbb{N}}$ similarly

  $\{C \cap A \mid C \in B_V\}$
- Sub spaces: $A \in P\lfloor V \rfloor$ $\quad A := (A, [B_V] \cap A)$

Def: Borel measurable functions $f : V_1 \to V_2$

- functions $f : \lfloor V_1 \rfloor \longrightarrow \lfloor V_2 \rfloor$
- inverse image preserves measurability:

$$f^{-1}[A] \in \mathcal{B}_{V_1} \Longleftarrow A \in \mathcal{B}_{V_2}$$

Examples

- $(+), (\cdot) : \mathbb{R}^2 \longrightarrow \mathbb{R}$
- $1{-}1$, $\sin : \mathbb{R} \to \mathbb{R}$
- any continuous function $f : \mathbb{R}^n \to \mathbb{R}^m$
- any function $f : \ulcorner X \urcorner \to V$

Category **Meas**

Objects : Measurable spaces

Morphisms: Measurable functions

Identities:

$id : V \to V$

Composition:

$$\frac{f : V_2 \to V_3 \qquad g : V_1 \to V_2}{f \circ g : V_1 \to V_3}$$

Meas Category

Products, coproducts/disjoint union, subspaces
Categorical limits, colimits, but:

<u>Thm</u> [Aumann '61] No σ-algebras $B_{B_{\mathbb{R}}}$, $B_{\mathbb{R}^{\mathbb{R}}}$ for measurable

membership predicate

$$(\ni) : (B_{\mathbb{R}}, B_{B_{\mathbb{R}}}) \times \mathbb{R} \longrightarrow \mathrm{Bool}$$

$$(U, r) \longmapsto [r \in U]$$

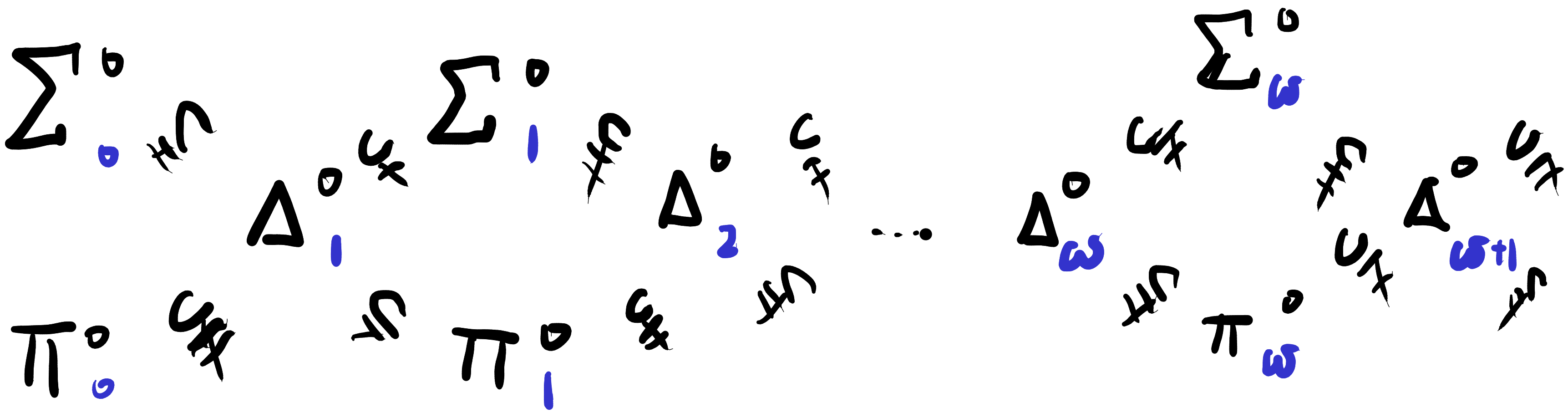
$$\mathrm{eval} : (\mathrm{Meas}(\mathbb{R}, \mathbb{R}), B_{\mathbb{R}^{\mathbb{R}}}) \times \mathbb{R} \to \mathbb{R}$$

$$(f, r) \mapsto f(r)$$

Questions? skip proof?

Proof (sketch):

Borel hierarchy:



Stabilises at $\Delta^0_{\omega_1} = \mathcal{B}(\Sigma^0_0) = \Delta^0_{\omega_1+1}$

rank $A := \min\{\alpha < \omega_1 \mid A \in \Delta^0_\alpha\}$

even for $B_{B_{\mathbb{R}}} = P(B_{\mathbb{R}})$

$$(\ni) : (B_{\mathbb{R}}, B_{B_{\mathbb{R}}}) \times \mathbb{R} \to \mathbb{R}$$

$$(U, r) \mapsto [r \in U]$$

$B_{V \times U} = B([B_V] \times [B_U])$

If measurable:

$$\alpha := \mathrm{rank}\left((\ni)^{-1}[\mathrm{true}]\right) < \omega_1$$

Take $A \in B_{\mathbb{R}}$, $\mathrm{rank}\, A > \alpha$

But:

$$\alpha < \mathrm{rank}\, A = \mathrm{rank}\left(\langle A, -\rangle^{-1}[(\ni)^{-1}[\mathrm{true}]]\right) \leq \mathrm{rank}\left((\ni)^{-1}[\mathrm{true}]\right) \leq \alpha$$

$\#$

More details in Ex. B

Sequential Higher-order Structure:

I Countable : $V^{\ulcorner I \urcorner} = \prod_{i \in I} V$

$\Longrightarrow$ Some higher-order structure in **Meas**:

Cauchy $\in B_{[-\infty,\infty]^{\mathbb{N}}}$

$$\mathrm{Cauchy} := \bigcap_{\varepsilon \in \mathbb{Q}^+} \bigcup_{k \in \mathbb{N}} \bigcap_{\substack{m,n \in \mathbb{N} \\ m,n \geq k}} \{ \vec{y} \in [-\infty,\infty]^{\mathbb{N}} \mid |y_m - y_n| < \varepsilon \}$$

lim sup $: [-\infty,\infty]^{\mathbb{N}} \to [-\infty,\infty]$ lim: Cauchy $\to \mathbb{R}$

Compose higher-order building blocks:

lim is measurable!

$$\mathrm{VanishingSeq}(\mathbb{R}) := \{ \vec{r} \in \mathbb{R}^{\mathbb{N}} \mid \lim_{n\to\infty} r_n = 0 \} \in B_{\mathbb{R}^{\mathbb{N}}}$$

$$\mathrm{approx}\_ : \mathrm{VanishingSeq}(\mathbb{R}^{+}) \times \mathbb{R} \longrightarrow \mathbb{Q}^{\mathbb{N}}$$

s.t.: $\left|(\mathrm{approx}_{\vec{\Delta}}\, r)_n - r\right| < \Delta_n$

Slogan: Measurable by Type!

Not all operations of interest fit:

Intrinsically higher-order!

$$\limsup : \left([-\infty,\infty]^{\mathbb{R}}\right)^{\mathbb{N}} \longrightarrow [-\infty,\infty]^{\mathbb{R}}$$

$$\limsup := \lambda \vec{f}.\lambda x. \limsup_{n\to\infty} f_n\, x$$

Want

Slogan: Measurable by Type!

But

For higher-order building-blocks, must

defer measurability proofs until we're

1st order again. ⇒ non-compositionality

Plan

Def: $V \in \mathbf{Meas}$ is **Standard Borel** when

$$V \cong A \qquad \text{for some } A \in \mathcal{B}_{\mathbb{R}}$$

the "good part" of **Meas** — the subcategory

$$\mathbf{Sbs} \hookrightarrow \mathbf{Meas}$$

Sbs include

- Discrete $\ulcorner I \urcorner$, I countable
- Countable products of Sbs:

$$\mathbb{R}^n, \mathbb{R}^{\mathbb{N}}, 2^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}$$

- Borel subspaces of Sbs:

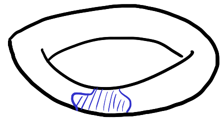
$\mathbb{I} := [0,1]$ $\qquad$ $\mathbb{R}^+ := (0,\infty)$ $\mathbb{R}_{\geq 0} := [0,\infty]$

- Countable coproducts of Sbs:

$\mathbb{W} := [0,\infty]$ $\qquad$ $\overline{\mathbb{R}} := [-\infty,\infty]$

Agenda

Slogan: Measurable by Type

- Borel sets 
- Qbs:
  def., constructions,
  partiality, type structure
- Measures & integration
- Random variable spaces
- Conditional expectation

Def: Quasi-Borel space $X = (\lfloor X \rfloor, R_X)$

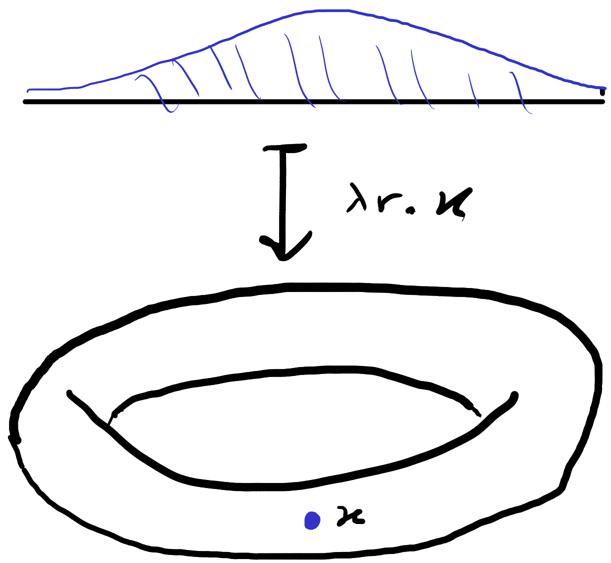
set "carrier"

set of functions $\alpha : \mathbb{R} \to \lfloor X \rfloor$ "random elements"

$R_X \subseteq \lfloor X \rfloor^{\lfloor \mathbb{R} \rfloor}$ closed under:

- Constants:

$$\frac{x \in \lfloor X \rfloor}{(\lambda r. x) \in R_X}$$



- precomposition:

- recombination

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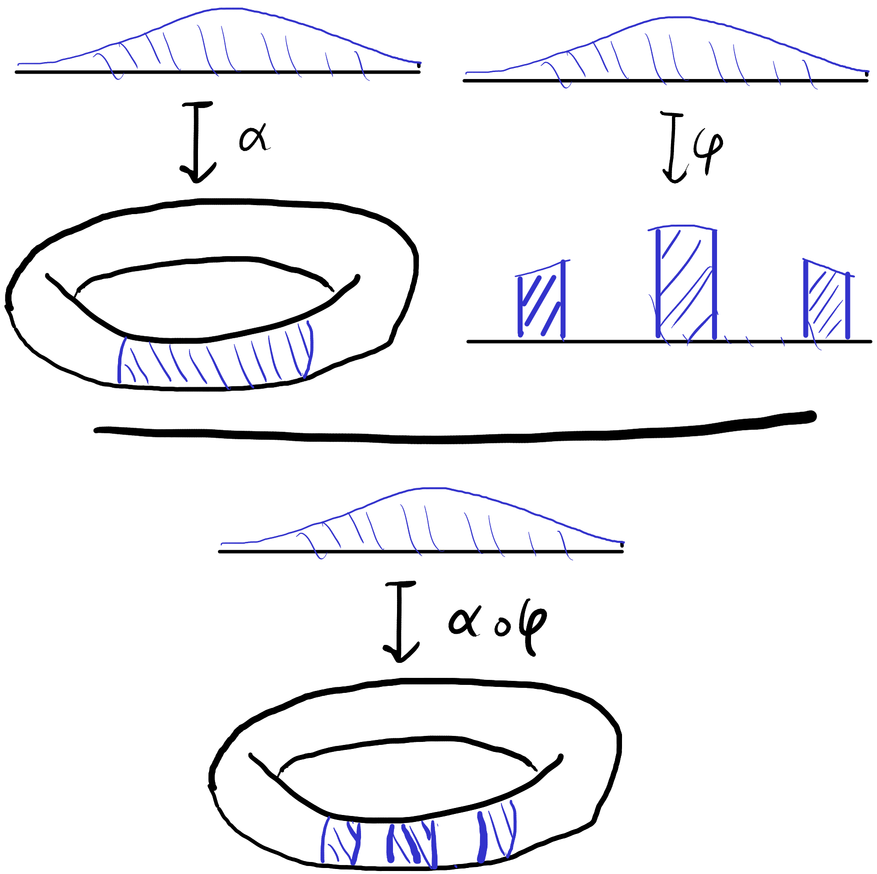
- precomposition:

$$\frac{\alpha \in R_X \qquad \varphi : \mathbb{R} \to \mathbb{R} \text{ in Sbs}}{\varphi \circ \alpha : \mathbb{R} \xrightarrow{\varphi} \mathbb{R} \xrightarrow{\alpha} \lfloor X \rfloor \in R_X}$$



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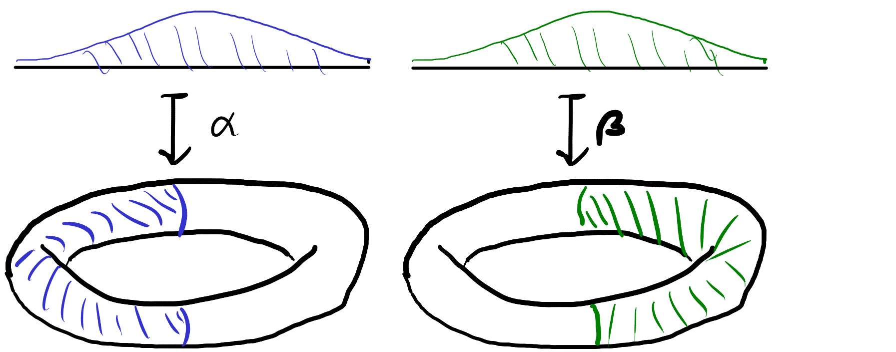
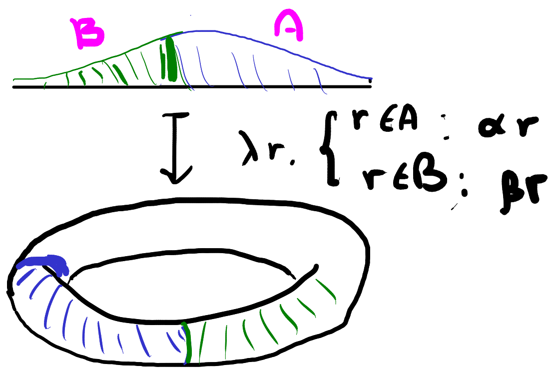
$$\frac{\vec{\alpha} \in R_X^{\mathbb{N}} \qquad \mathbb{R} = \biguplus_{n=0}^{\infty} \underbrace{A_n}_{\in B_{\mathbb{R}}}}{\lambda r. \begin{cases} \vdots \\ r \in A_n : \alpha_n r \\ \vdots \end{cases}}$$





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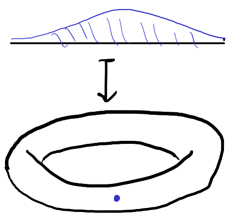
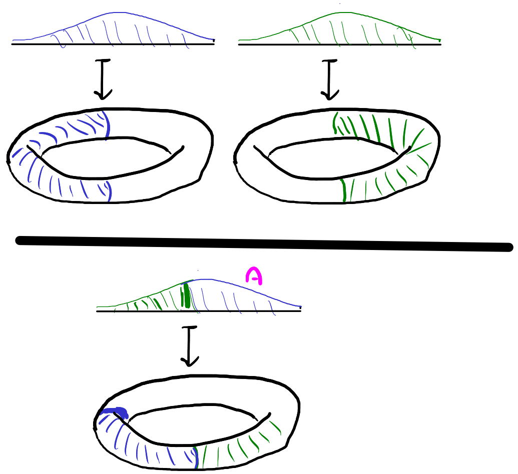
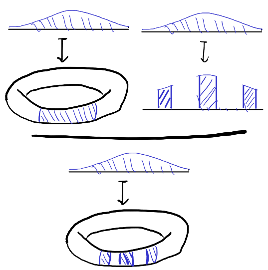
- constants: 

- recombination 

- precomposition: 

Examples

- $\mathbb{R} = (\lfloor\mathbb{R}\rfloor, \mathrm{Meas}(\mathbb{R},\mathbb{R}))$

  qbs underlying $\mathbb{R}$

- $X \in \mathrm{set}$, $\ulcorner X \urcorner^{\mathrm{Qbs}} := (X, \sigma\text{-simple}(\mathbb{R}, X))$

  recombination of constants

  $\lambda r. \begin{cases} \vdots \\ r \in A_n : x_n \\ \vdots \end{cases}$

  discrete qbs on X

- " $\lfloor X \rfloor_{\mathrm{Qbs}} := (X, X^{\lfloor\mathbb{R}\rfloor})$

  all funtions

  Indiscrete qbs on X

Qbs **morphism** $f : X \to Y$

- function $f : \lfloor X \rfloor \to \lfloor Y \rfloor$
- $$\frac{\mathbb{R} \xrightarrow{\alpha} \lfloor X \rfloor \in R_X}{\mathbb{R} \xrightarrow{\alpha} \lfloor X \rfloor \xrightarrow{f} \lfloor Y \rfloor \in R_Y}$$

Example

- **Constant** functions are qbs morphism
- **σ-simple** functions are qbs morphisms

Category **Qbs** $\Leftarrow$ - identity, composition