An introduction to statistical modelling semantics with higher-order measure theory

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Scottish Programming Languages and Verification (SPLV'22)
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Heriot-Watt University

THE UNIVERSITY of EDINBURGH informatics lfcs Laboratory for Foundations of Computer Science BayesCentre

supported by: THE ROYAL SOCIETY The Alan Turing Institute Facebook Research NCSC

Spaces Statistical Modelling needs:

Discrete finite

Bool: true false

Animal: cat, dog, Giraffe

Discrete Countable

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, String

Continuous

$\mathbb{R}^n$ $\mathbb{R}^{\mathbb{N}}$

$\mathcal{P}\mathbb{R}$,

Weight

standard Borel spaces

Measurable

Recent developments:

Discrete finite

Discrete Countable

Probabilistic Coherence spaces & Measurable Cones

[Ehrhard-Pagani-Tasson '18]

Boolean-Valued Models

[Bacci et al '18]

Regular ordered Banach

[Dahlqvist-Kozen '20]

Continuous

interval domains

Quasi-Borel Spaces

[Heunen et al. 17]

~~Measurable~~

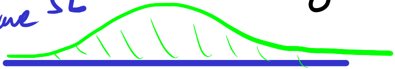
[Jia et al. '21]

This talk

Core idea

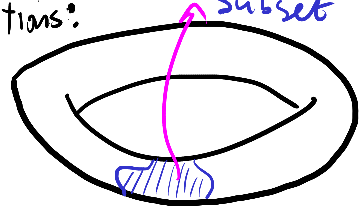
Measure Theory | **Qbs Theory**

sample space Ω

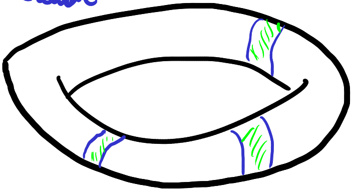


Primitive notions:

measurable subset



random element $\downarrow \alpha$



Derived notions:

random elements $\alpha : \Omega \to \text{Space}$

measure

measurable subsets

Conservative extensions:

concrete spaces we "observe"

Standard Borel spaces

Meas

Qbs

abstract spaces

(auxiliary?)

Wide topic:

Variations

Qbs, wQSs, [Vákár et al '19]

QMS, QUS, [Forré '21]

(w)Diff, wPap [Lew et al. '22]

[Vákár et al. '20-'21]

Applications

MC inference

design A

verification [Scibior et al. '18a+b]

Network Programming [Vandenbroucke-Schrijvers '19]

Semantics

name generation [Sabok et al. '21]

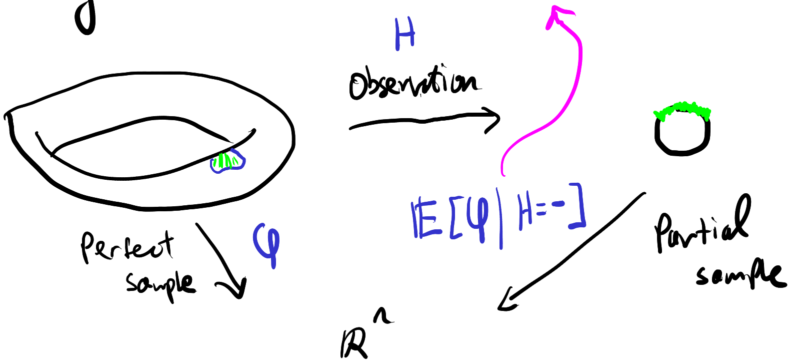
This course:

- Peek behind scenes
- Gain working knowledge

Theme: higher-order measure theory

demonstrated through

Kolmogorov's Conditional Expectation



Kolmogorov's Conditional Expectation

- naturally higher order: $\mathbb{R}^{\Omega} \to \mathbb{R}^{\Theta}$
- behind many modern Probability techniques:
  - existence of Radon-Nikodym derivatives & density
  - existence of disintegration
  - foundation of martingales & stochastic differential equations

Agenda

I
- Borel sets
- Qbs:
  def., constructions, partiality, type structure

II
- Measures & integration
- Random variable spaces
- Conditional expectation

Slogan:

Measurable by Type

NB:

- Exercise Sheets 
- #qbs on SPLS Zulip

Space: all possible states

Subset: all states of current interest

Measure: probability/weight/length assignment to

fine for discrete spaces

eg. $\{H, T\}^5$

HHTTH

$\frac{1}{32}$

Continuous Caveat:

Thm: no $\lambda : \mathcal{P}\mathbb{R} \to [0, \infty]$:

$\lambda(a, b) = b - a$

(generalises length)

$\lambda(r + A) = \lambda A$

(translation invariant)

$\lambda\left(\biguplus_{n=0}^{\infty} A_n\right) = \sum_{n=0}^{\infty} \lambda A_n$

σ-additive

Workaround: only measure well-behaved subsets

Def: The Borel subsets $B_{\mathbb{R}} \subseteq \mathbb{R}$:

- open intervals $(a,b) \in B_{\mathbb{R}}$

Closure under σ-algebra operations:

$$\frac{}{\emptyset \in B_{\mathbb{R}}} \qquad \frac{A \in B_{\mathbb{R}}}{A^c := \mathbb{R} \setminus A \in B} \qquad \frac{\vec{A} \in B_{\mathbb{R}}^{\mathbb{N}}}{\bigcup_{n=0}^{\infty} A_n \in B_{\mathbb{R}}}$$

empty set   complements   countable unions

Examples

discrete Countable: $\{r\} = \bigcap_{\varepsilon \in \mathbb{Q}^+} (r-\varepsilon, r+\varepsilon) \in \mathcal{B}_{\mathbb{R}}$

I countable $\Rightarrow$ $I = \bigcup_{r \in I} \{r\} \in \mathcal{B}_{\mathbb{R}}$

closed intervals: $[a,b] = (a,b) \cup \{a,b\}$

Non-examples?

More complicated: analytic, Lebesgue

Def: Measurable space $V = (\lfloor V \rfloor, B_V)$

Set (carrier)

family of subsets $B_V \subseteq P\lfloor V \rfloor$

Closed under σ-algebra operations:

$$\frac{}{\emptyset \in B_V} \qquad \frac{A \in B_V}{A^c := \lfloor V \rfloor \setminus A \in B_V} \qquad \frac{\vec{A} \in B_V^{\mathbb{N}}}{\bigcup_{n=0}^{\infty} A_n \in B_V}$$

empty set

complements

countable unions

Idea: Structure all spaces after the worst-case scenario

Examples

- Discrete spaces $\overset{\ulcorner\text{meas}\urcorner}{X} = (X, PX)$
- Euclidean spaces $\mathbb{R}^n$ — replace intervals with chests $\prod_{i=1}^{n} (a_i, b_i)$

  $\mathbb{R}^{\mathbb{N}}$ similarly

$\{C \cap A \mid C \in B_V\}$

- Subspaces: $A \in P\lfloor V \rfloor$ $\quad A := (A, [B_V] \cap A)$

Def: Borel measurable functions $f: V_1 \to V_2$

- functions $f : \lfloor V_1 \rfloor \longrightarrow \lfloor V_2 \rfloor$
- inverse image preserves measurability:

$$f^{-1}[A] \in \mathcal{B}_{V_1} \Longleftarrow A \in \mathcal{B}_{V_2}$$

Examples

- $(+), (\cdot) : \mathbb{R}^2 \longrightarrow \mathbb{R}$
- $|-|$, $\sin : \mathbb{R} \to \mathbb{R}$
- any continuous function $f : \mathbb{R}^n \to \mathbb{R}^m$
- any function $f : \text{`}X\text{'} \to V$

Category **Meas**

Objects: Measurable spaces

Morphisms: Measurable functions

Identities:

$id : V \to V$

Composition:

$$\frac{f : V_2 \to V_3 \qquad g : V_1 \to V_2}{f \circ g : V_1 \to V_3}$$

Meas Category

Products, coproducts/disjoint union, subspaces

Categorical limits, colimits, but:

Thm [Aumann '61] No σ-algebras $B_{B_{\mathbb{R}}}$, $B_{\mathbb{R}^{\mathbb{R}}}$ for measurable

membership predicate

$$(\ni) : (B_{\mathbb{R}}, B_{B_{\mathbb{R}}}) \times \mathbb{R} \longrightarrow \mathrm{Bool}$$

$$(U, r) \longmapsto [r \in U]$$

$$\mathrm{eval} : (\mathrm{Meas}(\mathbb{R}, \mathbb{R}), B_{\mathbb{R}^{\mathbb{R}}}) \times \mathbb{R} \to \mathbb{R}$$

$$(f, r) \mapsto f(r)$$

Questions? skip proof?

Proof (sketch):

Borel hierarchy:

$$\begin{tikzcd}[column sep=small]
\Sigma^0_0 \arrow[dr, phantom, "\subsetneq" sloped] & & \Sigma^0_1 \arrow[dr, phantom, "\subsetneq" sloped] & & & & \Sigma^0_\omega \arrow[dr, phantom, "\subsetneq" sloped] & & \\
& \Delta^0_1 \arrow[ur, phantom, "\subsetneq" sloped] \arrow[dr, phantom, "\subsetneq" sloped] & & \Delta^0_2 \arrow[r, phantom, "\subsetneq" sloped] & \cdots & \Delta^0_\omega \arrow[ur, phantom, "\subsetneq" sloped] \arrow[dr, phantom, "\subsetneq" sloped] & & \Delta^0_{\omega+1} \arrow[r, phantom, "\subsetneq" sloped] & {} \\
\Pi^0_0 \arrow[ur, phantom, "\subsetneq" sloped] & & \Pi^0_1 \arrow[ur, phantom, "\subsetneq" sloped] & & & & \Pi^0_\omega \arrow[ur, phantom, "\subsetneq" sloped] & &
\end{tikzcd}$$

stabilises at $\Delta^0_{\omega_1} = \mathcal{B}(\Sigma^0_0) = \Delta^0_{\omega_1+1}$

$$\operatorname{rank} A := \min\{\alpha < \omega_1 \mid A \in \Delta^0_\alpha\}$$

even for $B_{B_{\mathbb{R}}} = P(B_{\mathbb{R}})$

$$(\ni) : (B_{\mathbb{R}}, B_{B_{\mathbb{R}}}) \times \mathbb{R} \to \mathbb{R}$$

$$(U, r) \mapsto [r \in U]$$

$B_{V \times U} = B([B_V] \times [B_U])$

If measurable:

$\alpha := \mathrm{rank}\left((\ni)^{-1}[\mathrm{true}]\right) < \omega_1$

Take $A \in B_{\mathbb{R}}$, $\mathrm{rank}\, A > \alpha$

But:

$$\alpha < \mathrm{rank}\, A = \mathrm{rank}\left(\langle A, -\rangle^{-1}\left[(\ni)^{-1}[\mathrm{true}]\right]\right) \leq \mathrm{rank}\left((\ni)^{-1}[\mathrm{true}]\right) \leq \alpha$$

$\#$

More details in Ex. B

Sequential Higher-order structure:

I Countable: $V^{\ulcorner I \urcorner} = \prod_{i \in I} V$

$\Longrightarrow$ Some higher-order structure in **Meas**:

Cauchy $\in B_{[-\infty,\infty]^{\mathbb{N}}}$

$$\mathrm{Cauchy} := \bigcap_{\varepsilon \in \mathbb{Q}^+} \bigcup_{k \in \mathbb{N}} \bigcap_{\substack{m,n \in \mathbb{N} \\ m,n \geq k}} \left\{ \vec{y} \in [-\infty,\infty]^{\mathbb{N}} \,\middle|\, |y_m - y_n| < \varepsilon \right\}$$

lim sup $: [-\infty,\infty]^{\mathbb{N}} \to [-\infty,\infty]$ lim: Cauchy $\to \mathbb{R}$

Compose higher-order building blocks:

lim is measurable!

$$\mathrm{VanishingSeq}(\mathbb{R}) := \left\{ \vec{r} \in \mathbb{R}^{\mathbb{N}} \;\middle|\; \lim_{n\to\infty} r_n = 0 \right\} \in B_{\mathbb{R}^{\mathbb{N}}}$$

$$\mathrm{approx}\_ : \mathrm{VanishingSeq}(\mathbb{R}^+) \times \mathbb{R} \longrightarrow \mathbb{Q}^{\mathbb{N}}$$

$$\text{s.t.:} \quad \left| (\mathrm{approx}_{\vec{\Delta}}\, r)_n - r \right| < \Delta_n$$

Slogan: Measurable by Type!

Not all operations of interest fit:

Intrinsically higher-order!

$$\limsup : \left([-\infty,\infty]^{\mathbb{R}}\right)^{\mathbb{N}} \longrightarrow [-\infty,\infty]^{\mathbb{R}}$$

$$\limsup := \lambda \vec{f}.\lambda x. \limsup_{n\to\infty} f_n\, x$$

Want

Slogan: Measurable by Type !

But

For higher-order building-blocks, must

defer measurability proofs until we're

1^{st} order again. $\Rightarrow$ non-compositionality

Plan

Def: $V \in \mathbf{Meas}$ is **Standard Borel** when

$$V \cong A \quad \text{for some } A \in \mathcal{B}_{\mathbb{R}}$$

the "good part" of **Meas** — the subcategory

$$\mathbf{Sbs} \hookrightarrow \mathbf{Meas}$$

Sbs include

- Discrete $\ulcorner I \urcorner$, I countable
- Countable products of Sbs:

$$\mathbb{R}^n, \mathbb{R}^{\mathbb{N}}, 2^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}$$

- Borel subspaces of Sbs:

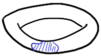
$$\mathbb{I} := [0,1] \qquad \mathbb{R}^+ := (0,\infty) \quad \mathbb{R}_{\geq 0} := [0,\infty]$$

- Countable coproducts of Sbs:

$$\mathbb{W} := [0,\infty] \qquad \overline{\mathbb{R}} := [-\infty,\infty]$$

Agenda

Slogan: Measurable by Type

- Borel sets 
- Qbs:
  def., constructions,
  partiality, type structure
- Measures & integration
- Random variable spaces
- Conditional expectation

Def: Quasi-Borel space $X = (\lfloor X \rfloor, R_X)$

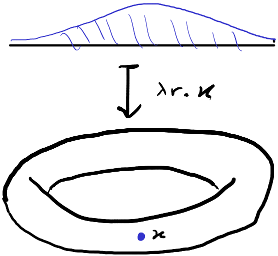
set "carrier"

set of functions $\alpha : \lfloor \mathbb{R} \rfloor \to \lfloor X \rfloor$ "random elements"

$R_X \subseteq \lfloor X \rfloor^{\lfloor \mathbb{R} \rfloor}$ closed under:

- Constants:

$$\frac{x \in \lfloor X \rfloor}{(\lambda r. x) \in R_X}$$



- precomposition:

- recombination

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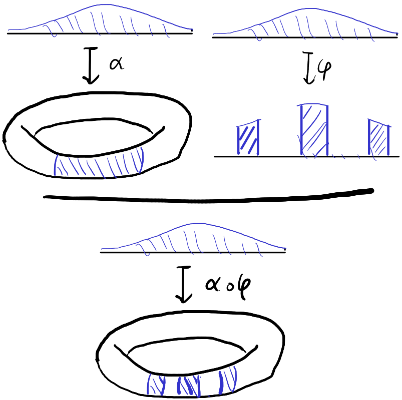
- precomposition:

$$\frac{\alpha \in R_X \qquad \varphi : \mathbb{R} \to \mathbb{R} \text{ in Sbs}}{\varphi \circ \alpha : \mathbb{R} \xrightarrow{\varphi} \mathbb{R} \xrightarrow{\alpha} \lfloor X \rfloor \in R_X}$$



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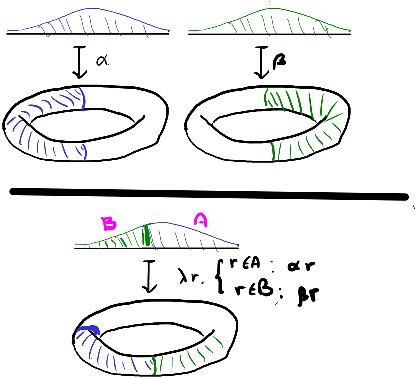
- recombination

$$\frac{\vec{\alpha} \in R_X^{\mathbb{N}} \qquad \mathbb{R} = \biguplus_{n=0}^{\infty} \underbrace{A_n}_{\in \mathcal{B}_{\mathbb{R}}}}{\lambda r. \begin{cases} \vdots \\ r \in A_n : \alpha_n r \\ \vdots \end{cases}}$$



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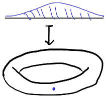
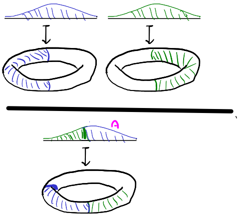
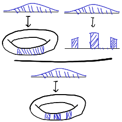
- Constants: 

- recombination 

- precomposition: 

Examples

recombination of constants

- $\mathbb{R} = (\lfloor\mathbb{R}\rfloor, \mathrm{Meas}(\mathbb{R},\mathbb{R}))$

  qbs underlying $\mathbb{R}$

$$\lambda r. \begin{cases} \vdots \\ r \in A_n : x_n \\ \vdots \end{cases}$$

- $X \in \mathrm{Set}$, $\ulcorner X \urcorner^{\mathrm{Qbs}} := (X, \sigma\text{-}\mathrm{Simple}(\mathbb{R}, X))$

  discrete qbs on X

- " $\lfloor X \rfloor_{\mathrm{Qbs}} := (X, X^{\lfloor\mathbb{R}\rfloor})$

  all funtions

  Indiscrete qbs on X

Qbs morphism $f : X \to Y$

- function $f : \lfloor X \rfloor \to \lfloor Y \rfloor$
- $$\frac{\mathbb{R} \xrightarrow{\alpha} \lfloor X \rfloor \in R_X}{\mathbb{R} \xrightarrow{\alpha} \lfloor X \rfloor \xrightarrow{f} \lfloor Y \rfloor \in R_Y}$$

Example

- Constant functions are qbs morphisms
- σ-simple functions are qbs morphisms

Category Qbs $\Leftarrow$ - identity, composition

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Recap: Semantic Foundation for Statistics & Probability

finite discrete spaces

Points/ states $\{H, T\}$

Events $\{H\}, \{T\}$

Probability $\frac{3}{4}$ $\frac{1}{4}$

but also: $\emptyset \mapsto 0$

$\{H, T\} \mapsto 1$

Recap: Semantic Foundation for Statistics & Probability

finite discrete spaces

Points/states $\{H, T\}$ $\{H, T\}^3$

more H than T

Events $\{H\}, \{T\}$ $\{HHT, HTH, THH, HHH\}$

$$= \bigcup_{i=1}^{3} \{H \cdots HTH \cdots H\} \cup \{H \cdots H\}$$

Probability $\frac{3}{4}$ $\frac{1}{4}$ $\sum_{i=1}^{3} \frac{3^2 \times 1}{4^3} + \frac{3^3}{4^3}$

but also: $\emptyset \mapsto 0$

$\{H, T\} \mapsto 1$

Countable discrete spaces

Points/states $\quad \mathrm{List}\{H, T\}$

more H than T

Events $\quad \bigcup_{n=0}^{\infty} \bigcup_{i=\lceil \frac{n}{2} \rceil}^{n} \bigcup_{\rho : \mathrm{Fin}\, i \rightarrowtail \mathrm{Fin}\, n} \left\{ F_1 \cdots F_n \text{ where } \begin{cases} i = \rho j : & F_i = H \\ \text{o.w} : & F_i = T \end{cases} \right\}$

list length

Countable discrete spaces

Points/states: List $\{H, T\}$

more H than T

Events:
$$\bigcup_{n=0}^{\infty} \bigcup_{i=\lceil \frac{n}{2} \rceil}^{n} \bigcup_{\rho : \mathrm{Fin}\, i \rightarrowtail \mathrm{Fin}\, n} \left\{ F_1 \cdots F_n \text{ where } \begin{cases} i = \rho j : & F_i := H \\ \text{o.w} : & F_i := T \end{cases} \right\}$$

list length

#H

positions of H

Probability:
$$\sum_{n=0}^{\infty} \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \sum_{\rho : \mathrm{Fin}\, i \rightarrowtail \mathrm{Fin}\, n} \frac{999}{1000^n} \cdot \frac{3^i \times 1^{n-i}}{4^n}$$

Continuous spaces

Points/ states $\mathbb{R}$

Events Borel subsets $\mathcal{B}_{\mathbb{R}}$

Probability $E \mapsto \int_E dx \frac{1}{\sqrt{2\pi}} e^{-2(x-1)^2} + \sum_{\substack{n=1 \\ 2n \in E}}^{\infty} \frac{1}{2^n}$

density w.r.t. measure $w : \text{points} \to [0,\infty]$

$$\lambda + \sum_{n=1}^{\infty} \delta_{2n}$$

Lebesgue ↳ Dirac

Continuous spaces

Points/ states $\mathbb{R}$

$\{H, T\}^{\mathbb{N}}$

Events Borel subsets $\mathcal{B}_{\mathbb{R}}$

σ-algebra generated by "open cylinders"

e.g. $HTHT\# \{H,T\}^{\mathbb{N}} \mapsto \frac{3^2 \times 1^2}{4}$

Probability $E \mapsto \int_E dx \frac{1}{\sqrt{2\pi}} e^{-2(x-1)^2} + \sum_{\substack{n=1 \\ 2n \in E}}^{\infty} \frac{1}{2^n}$

density w.r.t. measure $w : \text{points} \to [0, \infty]$

$\underbrace{\lambda}_{\text{Lebesgue}} + \sum_{n=1}^{\infty} \underbrace{\delta_{2n}}_{\text{Dirac}}$

Haar measure

Discrete finite

Discrete Countable

Continuous

Quasi-Borel Spaces

~~Measurable~~

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see CIRM videos
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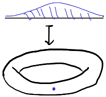
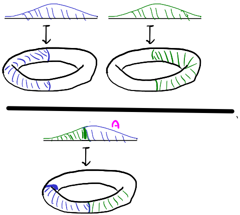
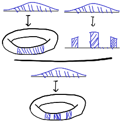
- constants: 

- recombination 

- precomposition: 

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discrete qbs on X

- " $\lfloor X \rfloor_{\mathrm{Qbs}} := (X, X^{\lfloor\mathbb{R}\rfloor})$

all functions

Indiscrete qbs on X

Useful adjunctions:

$$
\begin{tikzcd}
\mathbf{Meas} \arrow[d, "\llcorner - \lrcorner_{Qbs}"] \\
\mathbf{Qbs} \arrow[u, bend left, "\ulcorner - \urcorner^{Meas}"] \arrow[d, "\llcorner \cdot \lrcorner"] \\
\mathbf{Set} \arrow[u, bend left, "\ulcorner - \urcorner^{Qbs}"] \arrow[u, bend right, "\llcorner - \lrcorner_{Qbs}"']
\end{tikzcd}
$$

$\ulcorner - \urcorner^{Meas} \dashv \llcorner - \lrcorner_{Qbs}$

$\ulcorner - \urcorner^{Qbs} \dashv \llcorner \cdot \lrcorner \dashv \llcorner - \lrcorner_{Qbs}$

$$\llcorner V \lrcorner_{Qbs} := (\llcorner V \lrcorner, \mathbf{Meas}(\mathbb{R}, V)) \quad (V \in \mathbf{Meas})$$

$$\ulcorner X \urcorner^{Meas} := \left\{ A \subseteq \llcorner X \lrcorner \;\middle|\; \forall \alpha \in R_X .\ \alpha^{-1}[A] \in \mathcal{B}_{\mathbb{R}} \right\}$$

- limits (products, subspaces) and colimits (coproducts, quotients) as in Set
- Slogan: every measurable space is carried by a qbs

Example

Product $(X \times Y, \pi_1, \pi_2)$:

necessarily!

- $\lfloor X \times Y \rfloor = \lfloor X \rfloor \times \lfloor Y \rfloor$
- $R_{X \times Y} = \{ \lambda r. (\alpha r, \beta r) \mid \alpha \in R_X, \beta \in R_Y \}$

correlated random elements

rest of structure as in **Set**.

Function Spaces

Straightforward!

- $\lfloor Y^X \rfloor := \mathbf{Qbs}(X, Y)$
- $R_{Y^X} := \mathrm{uncurry}[\mathbf{Qbs}(\mathbb{R} \times X, Y)]$
  $= \{\alpha : \mathbb{R} \to \lfloor Y^X \rfloor \mid \lambda(r, x).\, \alpha\, r\, x : \mathbb{R} \times X \to Y\}$
- $\mathrm{eval} : Y^X \times X \to Y$
  $\mathrm{eval}(f, x) := f\, x$

Meas vs Qbs

By generalities:

σ-algebra on $\mathbf{Meas}(\mathbb{R},\mathbb{R})$

$$\begin{tikzcd}
\ulcorner \mathbb{R}^{\mathbb{R}} \urcorner^{\mathbf{Meas}} \times \mathbb{R} \arrow[r] \arrow[rr, bend right, "\ulcorner \mathrm{eval} \urcorner^{\mathbf{Meas}}"'] & \ulcorner \mathbb{R}^{\mathbb{R}} \urcorner^{\mathbf{Meas}} \times \mathbb{R} \arrow[r, dashed, "/" marking] & \ulcorner \mathbb{R} \urcorner = \mathbb{R}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{Meas} \arrow[d, "L_{\mathbf{Qbs}}"] \arrow[d, phantom, shift right=6, "\dashv"] \\
\mathbf{Qbs} \arrow[u, bend left=60, "\ulcorner - \urcorner^{\mathbf{Meas}}"]
\end{tikzcd}$$

No factorisation by Aumann's Theorem.

(So $\ulcorner \mathbb{R}^{\mathbb{R}} \times \mathbb{R} \urcorner \not\cong \ulcorner \mathbb{R}^{\mathbb{R}} \urcorner \times \ulcorner \mathbb{R} \urcorner$)

Random element Space

$$R_X := X^{\mathbb{R}} \quad \text{since} \quad \lfloor X^{\mathbb{R}} \rfloor = R_X \text{ as sets.}$$

Why?

$(\subseteq)$ $\alpha \in \lfloor X \rfloor^{\mathbb{R}} \Rightarrow \alpha : \mathbb{R} \to X$ in Qbs.

$\mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ measurable $\Rightarrow \mathrm{id} \in R_{\mathbb{R}}$

$\Rightarrow \alpha = \alpha \circ \mathrm{id} \in R_X$

Precomposition

$(\supseteq)$ $\alpha \in R_X \Rightarrow \forall \varphi \in R_{\mathbb{R}} = \mathrm{Meas}(\mathbb{R}, \mathbb{R}).\ \alpha \circ \varphi \in R_X \Rightarrow \alpha : \mathbb{R} \to X$

$\Rightarrow \alpha \in \lfloor X^{\mathbb{R}} \rfloor$

Subspaces

For $X \in \mathbf{Qbs}$, $A \subseteq \lfloor X \rfloor$ set:

$$R_A := \{ \alpha : \mathbb{R} \to A \mid \alpha \in R_X \}$$

Then $A = (A, R_A)$ is the subspace qbs

We write $A \hookrightarrow X$

Borel subspaces ensemble

The σ-algebra $\mathcal{B}_X := \{ A \subseteq \lfloor X \rfloor \mid \forall \alpha \in R_X .\ \alpha^{-1}[A] \in \mathcal{B}_{\mathbb{R}} \}$

internalises as $B_X = 2^X$, the qbs of

Borel subsets.

($\lfloor \mathcal{B}_{(\mathcal{B}_{\mathbb{R}})} \rfloor$ are the Borel-on-Borel sets from descriptive set theory.
cf. [Sabok et al. '21])

Standard Borel spaces

Def: A qbs S is standard Borel when

$$S \cong A \quad \text{for some } A \in \mathcal{B}_{\mathbb{R}}$$

$$\begin{tikzcd}
\mathbf{Meas} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathbf{Qbs} \arrow[l, bend left] \\
\mathbf{Sbs} \arrow[u, hook] \arrow[r, bend left] \arrow[r, phantom, "\cong"] & \mathbf{Sbs} \arrow[u, hook] \arrow[l, bend left]
\end{tikzcd}$$

Slogan: Qbs conservative extension of Sbs

Example $C_0 := \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous} \} \hookrightarrow \mathbb{R}^{\mathbb{R}}$

C_0 is sbs. (Well-known!)

Proof:

$C_0' \in B_{\mathbb{R}^{\mathbb{Q}}}$ sbs!

$$C_0' := \left\{ g \in \mathbb{R}^{\mathbb{Q}} \;\middle|\; \begin{array}{l} \forall a,b \in \mathbb{Q}, \varepsilon \in \mathbb{Q}^+ \\ \exists \delta \in \mathbb{Q}^+ \; \forall p,q \in \mathbb{Q}^+ \cap [a,b]. \\ |p-q| < \delta \Rightarrow |gp - gq| < \varepsilon \end{array} \right\}$$

on closed intervals (= compact intervals)

Continuity $\Updownarrow$ uniform continuity

Borel measurable by type checks

then $C_0 \cong C_0' \in B_{\mathbb{R}^{\mathbb{Q}}}$:

$$\begin{array}{ll} C_0 \to C_0' & C_0' \to C_0 \\ \varphi \mapsto \varphi|_{\mathbb{Q}} & \varphi \mapsto \lambda r. \lim_{n\to\infty} g(\text{approx}_{(\frac{1}{m})_{m\in\mathbb{N}}} r)_n \end{array}$$

Example (ctd)

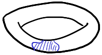
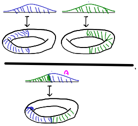
C_0 is sbs, and $eval : C_0 \times \mathbb{R} \to \mathbb{R}$ is measurable.

Avoids:

- constructing complete separable metrics
- proving that evaluation is measurable w.r.t. metric σ-algebra.

Agenda

Slogan: Measurable by Type

- Borel sets 
- Qbs:
  - def., constructions, 
  - partiality, type structure
- Measures & integration
- Random variable spaces
- Conditional expectation

Partiality cf. [Vákár et al. '19]

A **Borel embedding** $e : X \hookrightarrow Y$

- injective function $e : \lfloor X \rfloor \rightarrowtail \lfloor Y \rfloor$
- its image is Borel: $e[\lfloor X \rfloor] \in B_Y$
- e is **Strong**: $\alpha \in R_X \iff e \circ \alpha \in R_Y$

Examples

- $\mathbb{1} \hookrightarrow \mathbb{2}$
- S is sbs $\Leftrightarrow \exists\, S \hookrightarrow \mathbb{R}$

Non-examples $\rightsquigarrow$ [Sabok et al. '21]

- $\{A \in \mathcal{B}_{\mathbb{R}} \mid A \neq \emptyset\} \hookrightarrow \mathcal{B}_{\mathbb{R}}$

- $\{(A_1, A_2) \in \mathcal{B}_{\mathbb{R}}^2 \mid A \subseteq B\} \hookrightarrow \mathcal{B}_{\mathbb{R}}^2$

- $\{A \in \mathcal{B}_{\mathbb{R}} \mid A \text{ open}\} \hookrightarrow \mathcal{B}_{\mathbb{R}}$

Def: A partial map $f : X \rightharpoonup Y$ is a morphism

$$f : X \to Y \amalg \{\bot\}$$

Its domain of definition

$$\begin{tikzcd} \mathrm{Dom}\, f := \{x \mid fx \neq \bot\} \arrow[d, hook] \\ X \end{tikzcd}$$

Partial hom-sets are ordered:

for $f, g : X \rightharpoonup Y$ $\quad f \leq g$ when $\forall x.\ fx \neq \bot \Rightarrow gx = fx$.

[Cockett-Lack '06]

A model of restriction categories / axiomatic domain theory

[Fiore-Plotkin '94] Borel embeddings are the admissible monos

Type Structure

Simple types denote spaces

E.g.: $A \times B$ $\quad B^A$ $\quad B_A$

Dependent types denote spaces-in-context

$\Gamma \vdash A$

$$\begin{array}{c} [\![\Gamma \vdash A]\!] \\ \big\downarrow \mathrm{dep} \\ [\![\Gamma]\!] \end{array}$$

Dependent types denote spaces-in-Context

$\Gamma \vdash$ Context

$\Gamma \vdash A$

type in context

$$\llbracket \Gamma \vdash A \rrbracket \xrightarrow{\text{dep}} \llbracket \Gamma \rrbracket$$

Space in Context

Context space

assigns environment

E.g.:

$$A \longrightarrow \mathbb{1}$$

simple types

$$\llbracket U : \mathbb{B}_A \vdash U \rrbracket = \{(U, a) \in \mathbb{B}_A \times A \mid a \in U\} \xrightarrow{\pi_1} \mathbb{B}_A$$

sbs decoder

Context extension

$$\frac{\Gamma \vdash A}{\Gamma, a : A \vdash}$$

$$\begin{array}{c} [\![\Gamma \vdash A]\!] \\ \Big\downarrow \mathrm{dep} \\ [\![\Gamma]\!] \end{array} \qquad [\![\Gamma, a : A]\!] := [\![\Gamma \vdash A]\!]$$

Substitution

$\Gamma \vdash \sigma : \Delta$ $\qquad [\![\sigma]\!] : [\![\Gamma]\!] \longrightarrow [\![\Delta]\!]$

E.g. weakening $\qquad \Gamma, a : A \vdash \mathrm{wkn} : \Gamma$

$$[\![\Gamma, a : A]\!] := [\![\Gamma \vdash A]\!] \xrightarrow[\mathrm{dep}]{\mathrm{wkn}} [\![\Gamma]\!]$$

Action of substitution on types

$$\begin{tikzcd}
{[\![\Gamma \vdash A[\sigma]]\!]} \\[-20pt]
\rotatebox{90}{=} \\[-20pt]
\{(\gamma, a) \in [\![\Gamma]\!] \times [\![A]\!] \mid \mathrm{dep}\, a = \gamma\} \arrow[r, "\pi_2"] \arrow[d, "\mathrm{dep} := \pi_1"'] \arrow[dr, phantom, "\lrcorner" very near start] & {[\![\Delta \vdash A]\!]} \arrow[d, "\mathrm{dep}"] \\
{[\![\Gamma]\!]} \arrow[r, "\sigma"] & \Delta
\end{tikzcd}$$

E.g.

$$\begin{tikzcd}
{[\![x : A \vdash B[\mathrm{wkn}]]\!] := A \times B} \arrow[r, "\pi_2"] \arrow[d, "\mathrm{dep} := \pi_1"'] \arrow[dr, phantom, "\lrcorner" very near start] & B \arrow[d] \\
x : A \arrow[r, "\langle\rangle"'] & \mathbb{1}
\end{tikzcd}$$

simple type

Terms: sections

$$\begin{tikzcd}
\llbracket \Gamma \rrbracket \arrow[r, "\llbracket \Gamma \vdash M : A \rrbracket"] \arrow[dr, equal, "="'] & \llbracket \Gamma \vdash A \rrbracket \arrow[d, "dep"] \\
 & \llbracket \Gamma \rrbracket
\end{tikzcd}$$

e.g.

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\llbracket x : \mathbb{R} \vdash {[x, \infty)} : \mathcal{B}_{\mathbb{R}}{[wkn]} \rrbracket"] \arrow[dr, equal, "="'] & \mathbb{R} \times \mathcal{B}_{\mathbb{R}} \arrow[d, "\pi_1"] \\
 & \mathbb{R}
\end{tikzcd}$$

E.g. Variables: $\llbracket \Gamma, a : A \vdash a : A \rrbracket$

$$\begin{tikzcd}
\llbracket \Gamma, a : A \rrbracket \arrow[r, "\langle id, dep_{\Gamma \vdash A} \rangle"'] \arrow[dr, equal, "="'] & \llbracket \Gamma, a : A \vdash A[wkn] \rrbracket \arrow[d, "dep"] \\
 & \llbracket \Gamma, a : A \rrbracket
\end{tikzcd}$$

Exercise:
action of substitution
$M[\sigma]$

Dependent Pairs

$$\frac{\Gamma, a:A \vdash B}{\Gamma \vdash \coprod_{a \in A} B}$$

$$[\![\coprod_{a\in A} A]\!] := [\![\Gamma, a:A \vdash B]\!]$$

$$\begin{tikzcd}
{[\![\coprod_{a\in A} A]\!]} \arrow[ddr, bend right, "\mathrm{dep}_{\coprod}"'] \arrow[r, phantom, ":="] & {[\![\Gamma, a:A \vdash B]\!]} \arrow[d, "\mathrm{dep}_B"] \\
& {[\![\Gamma, a:A]\!] = [\![\Gamma \vdash A]\!]} \arrow[d] \\
& {[\![\Gamma]\!]}
\end{tikzcd}$$

Dependent Products

$$\frac{\Gamma, a : A \vdash B}{\Gamma \vdash \prod_{a \in A} B}$$

aha: $(a:A) \to B$

$$\lfloor [\![\Gamma \vdash \prod_{a \in A} B]\!] \rfloor :=$$

$$\left\{ (r_0, f : \{ a \in [\![A]\!] \mid \mathrm{dep}\, a = r_0 \} \to [\![\Gamma, a : A \vdash B]\!]) \middle| \forall a \in [\![\Gamma, a : A]\!] .\ \mathrm{dep}\, a = r_0 \Rightarrow \mathrm{dep}(f\, a) = a \right\}$$

Exercise: find the random elements.

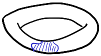
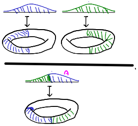
Example $(\Omega \in \mathbf{Qbs})$

$$\mathrm{Converging} \hookrightarrow \left([-\infty,\infty]^{\Omega}\right)^{\mathbb{N}}$$

$$\mathrm{Converging} := \prod_{\omega\in\Omega} \left\{ \vec{f} \;\middle|\; \exists \lim_{n\to\infty} f_n \omega \right\}$$

Agenda

Slogan: Measurable by Type

- Borel sets 
- Qbs: 
  def., constructions,
  partiality, refinement $\to$, $\Rightarrow$, $\amalg$, Π
- Measures & integration
- Random variable spaces
- Conditional expectation

Def: A measure μ over $\mathbb{R}$ is a function

$$\mu : B_{\mathbb{R}} \to W := [0, \infty]$$

s.t. - $\mu \emptyset = 0$

$$- \quad \frac{\vec{A} \in B_{\mathbb{R}}^{\mathbb{N}} \qquad \begin{array}{c} A_n \cap A_m = \emptyset \\ (n \neq m) \end{array}}{\mu\left(\bigcup_{n=0}^{\infty} A_n\right) = \sum_{n=0}^{\infty} \mu A_n}$$

For measurable spaces, replace $\mathbb{R}$ with V

We write $\lfloor G V \rfloor$ for the set of measures on V

For qbs X, take $\lfloor G \ulcorner \overset{\mathrm{Meas}}{X} \urcorner \rfloor$

The unrestricted Giry spaces

Equip $\lfloor GV \rfloor$ with two qbs structures:

$$R_{GV} := \{\alpha : \mathbb{R} \to GV \mid \forall A \in B_V.\ \lambda r.\ \alpha(r, A) : \mathbb{R} \to W\}$$

↪ α is a kernel.

$$GV \hookrightarrow W^{B_X}$$

- Fewer random elements
- Lebesgue integral measurable in both arguments. (upcoming)

Farewell Meas

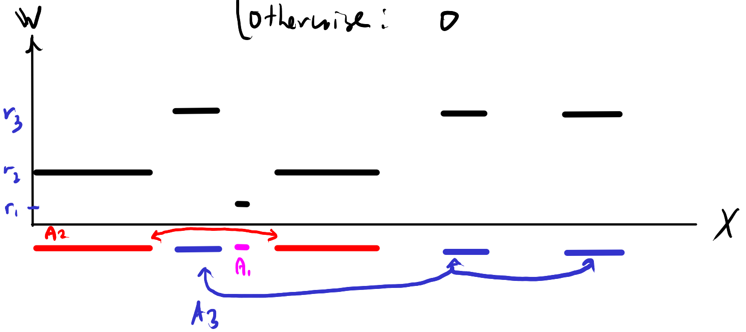
Now on:

1. All spaces are quasi-Borel
2. "measurable function" means qbs morphism!

Def: Simple function $\varphi : X \to W$ when

$\exists\, n \in \mathbb{N},\ \vec{A} \in B_X^n,\ A_i \cap A_j = \phi\ (i \neq j),\ \vec{r} \in W$ s.t.

$$\varphi x = \begin{cases} \vdots & \\ x \in A_i : & r_i \\ \vdots & \\ \text{otherwise}: & 0 \end{cases}$$



Encode into a space:

$$\mathrm{SimpleCode} := \coprod_{n \in \mathbb{N}} B_X^n \times W^n$$

$$\mathrm{Simple} := \{ f \in W^X \mid f \text{ simple} \} \hookrightarrow W^X$$

and define an interpretation:

$$[\![-]\!] : \mathrm{SimpleCode} \twoheadrightarrow \mathrm{Simple}$$

$$[\![(n, \vec{A}, \vec{r})]\!] := \sum_{i=1}^{n} r_i \cdot [- \in A_i]$$

↳ characteristic function for A_i

Lemma: $f : X \to W$ is measurable

remember! 90s morphism!

iff $f = \lim_{n\to\infty} f_n$ for some monotone sequence

$\vec{f} \in \mathrm{Simple}$.

Moreover, we have measurable such choice:

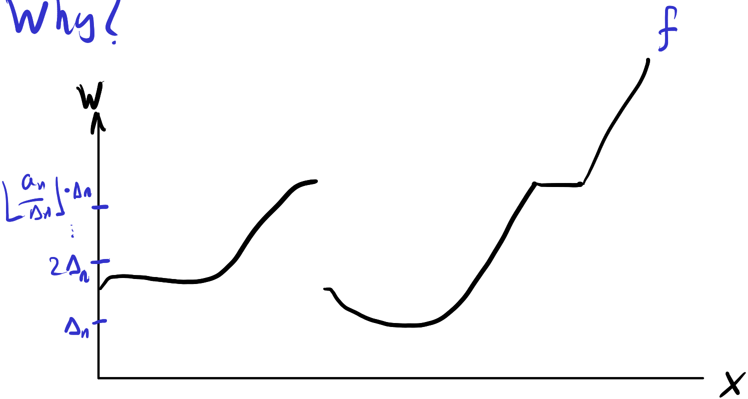
Simple Approx :

$$\{\vec{\Delta} \in \mathbb{R}^+ \mid \Delta_n \to 0\} \times \{\vec{a} \in W^{\mathbb{N}} \mid \vec{a} \text{ monotone}, a_n \to \infty\} \times W^X \to \mathrm{SimpleCode}$$

rate of convergence

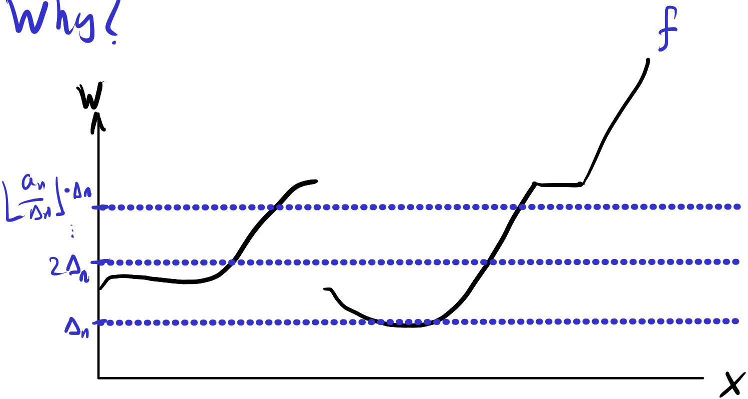
range of approximation

Why?



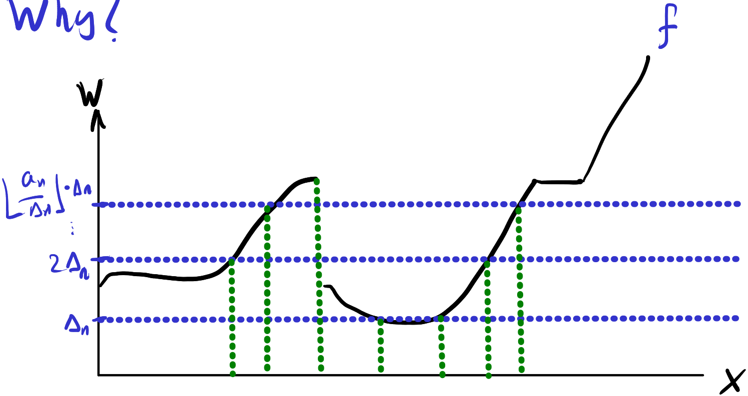
f
W
X
$\lfloor \frac{a_n}{\Delta_n} \rfloor \cdot \Delta_n$
$\vdots$
$2\Delta_n$
Δ_n

Why?



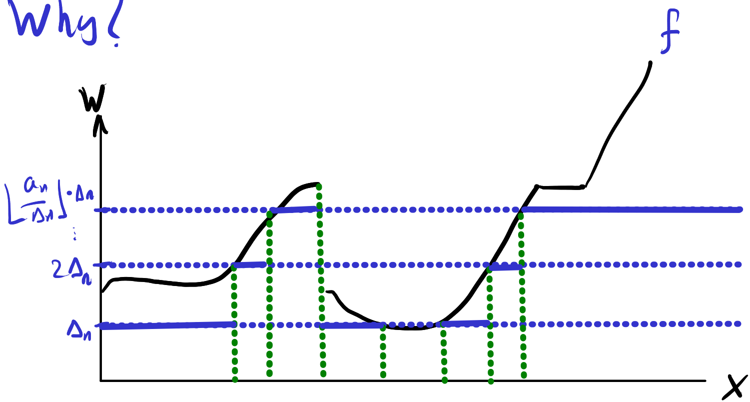
f
W
$\lfloor \frac{a_n}{\Delta_n} \rfloor \cdot \Delta_n$
$2\Delta_n$
Δ_n
X

Why?

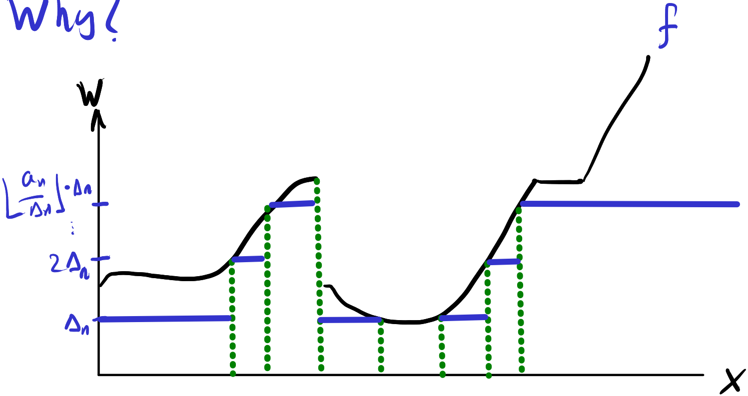


f
W
$\lfloor \frac{a_n}{\Delta_n} \rfloor \cdot \Delta_n$
$2\Delta_n$
Δ_n
X

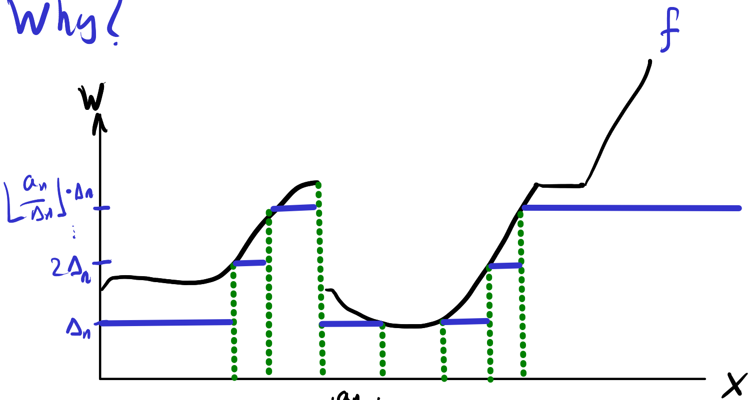
Why?


f
W
$\lfloor \frac{a_n}{\Delta_n} \rfloor \cdot \Delta_n$
$2\Delta_n$
Δ_n
X

Why?


f
W
$\lfloor \frac{a_n}{\Delta_n} \rfloor \cdot \Delta_n$
$2\Delta_n$
Δ_n
X

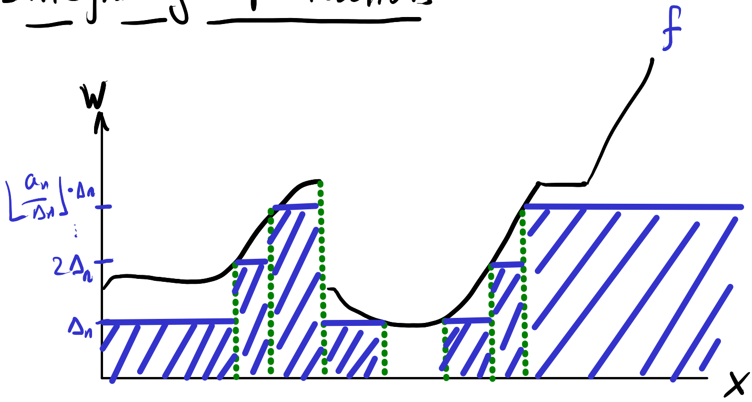
Why?



$$[\![\mathrm{SimpleApprox}_{\vec{\Delta},\vec{a}}\, f]\!] := \sum_{i=1}^{\lfloor \frac{a_n}{\Delta_n} \rfloor} i\cdot\Delta_n \left[i\cdot\Delta_n \le f < (i+1)\Delta_n \right]$$

$$+ \left\lfloor \frac{a_n}{\Delta_n} \right\rfloor \Delta_n \cdot \left[f \geq \left\lfloor \frac{a_n}{\Delta_n} \right\rfloor \cdot \Delta_n \right] \in \mathrm{Simple}$$

Integrating Simple Functions



$$\int : GX \times \mathrm{SimpleCode} \to W$$

$$\int \mu\,(n, \vec{A}, \vec{r}) := \sum_{I \subseteq \{1,\ldots,n\}} \left(\sum_{i \in I} r_i\right) \cdot \mu\left(\bigcap_{i \in I} A_i \setminus \bigcup_{i \notin I} A_i\right)$$

Integration

Properly higher-order operation

$$\int : GX \times \mathbb{W}^X \longrightarrow \mathbb{W}$$

$$\int \mu f := \sup \left\{ \int \mu \varphi \;\middle|\; \varphi \in \mathrm{Simple},\ \varphi \leq f \right\}$$

$$= \lim_{n \to \infty} \int \mu \left(\mathrm{SimpleApprox}_{\vec{\Delta}, \vec{a}}\, f\right)_n$$

measurable by type

we also write $\int \mu(dx) t$ for $\int \mu (\lambda x. t)$

for $\frac{a_n}{\Delta_n} \to 0$, e.g. $\Delta_n = \frac{1}{2^n}$ $a_n = n$.

resolution

The unrestricted Giry Strong Monad

Unlike the unrestricted Giry on Meas.

Dirac:

$$\delta : X \to GX$$

$$x \mapsto \lambda A. \begin{cases} x \in A : & 1 \\ x \notin A : & 0 \end{cases}$$

but: non-commutative

(Fubini fails, just like in Meas)

Kleisli extension / Kock integral:

$$\oint : GX \times GY^{X} \longrightarrow GY$$

$$\oint \mu f := \lambda A. \int \mu(dx)\, f\, x\, A$$

Randomisable measures monad

$D \rightarrowtail G$

$$\lambda A. \int_{\mathrm{Dom}\,\alpha} \lambda\ \alpha^{-1}[A]$$

$$\lfloor DX \rfloor := \{ \alpha_* \lambda \mid \alpha : \mathbb{R} \to X \}$$

Lebesgue measure

$$M_{DX} := \{ \lambda x. (\alpha x)_* \lambda \mid \alpha : \mathbb{R} \times \mathbb{R} \to X \}$$

D is commutative (Fubini's Theorem)

$\mu \in DX$, $\nu \in DY$:

$$\oint \mu(dx) \oint \nu(dy)\, \delta_{(x,y)} = \oint \nu(dy) \oint \mu(dx)\, \delta_{(x,y)} =: \mu \otimes \nu$$

Models Kock's Synthetic measure theory [Kock'12, Ścibior et al.'18]

Distribution Submonads

A measure space

$$\Omega = (\Omega, \mu)$$

is a qbs Ω with $\mu \in DX$.

Similarly: finite measure space

- (sub) probability space.

$$\begin{array}{c}
PX := \{\mu \in DX \mid \mu X = 1\} \\
\big\downarrow \\
P_{\leq 1}X := \{\mu \in DX \mid \mu X \leq 1\} \\
\big\downarrow \\
P_{<\infty}X := \{\mu \in DX \mid \mu X < \infty\} \\
\big\downarrow \\
DX
\end{array}$$

Thm: For sbs S, $PS, D_{\leq 1}S, D_{<\infty}S \in Sbs$

and agree with their counterparts on **Meas**.

$$[DS] = \{\mu \mid \mu \text{ s-finite}\}$$

See [Staton'16]

$$R_{DS} = \{ k : \mathbb{R} \to GD \mid k \text{ s-finite kernel} \}$$

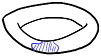
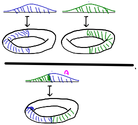
Open: Is there a counterpart to D in **Meas**?

More modestly, is $DS \in Sbs$?

(Hypothesis: no)

Agenda

Slogan: Measurable by Type

- Borel sets 
- Qbs:
  def., constructions, 
  partiality, refinement $\to, \rightharpoonup, \coprod, \prod$
- Measures & integration $\delta, \int, \fint, D, \otimes, P$
- Random variable spaces
- Conditional expectation

Random variable: $\xi : \Omega \to \Theta \hookrightarrow \overline{\mathbb{R}}$

Θ:

$$
\begin{tikzcd}
 & & & \mathbb{W} \arrow[rd, hook] & \\
\mathbb{I} \arrow[r, hook] & \mathbb{R}_{\geq 0} \arrow[rru, hook] \arrow[rr, hook] & & \mathbb{R} \arrow[r, hook] & \overline{\mathbb{R}} \\
 & \mathbb{R}^{+} \arrow[u, hook] & & &
\end{tikzcd}
$$

$\mathbb{I} = [0,1]$, $\mathbb{R}_{\geq 0} = [0,\infty)$, $\mathbb{W} = [0,\infty]$, $\overline{\mathbb{R}} = [-\infty,\infty]$, $\mathbb{R} = (-\infty,\infty)$, $\mathbb{R}^{+} = (0,\infty)$

- Θ^{Ω} is a space
- $\mathbb{R}^{\Omega}$ measurable vector space:

$$\alpha\xi + \zeta := \lambda\omega.\ \alpha\cdot\xi\omega + \zeta\omega$$

- $\mathbb{W}^{\Omega}$ measurable σ-semi-module for $\mathbb{W}$:

$$\sum_{n=0}^{\infty} \alpha_n \xi_n := \lambda\omega.\ \sum_{n=0}^{\infty} \alpha_n\cdot\xi_n\omega$$

$\Pr_- : P\Omega \times B_\Omega \to W$

$\Pr_\lambda A := \mathrm{eval}(\lambda, A) = \lambda A$

Probability Space $\Omega = (\Omega, \lambda_\Omega)$

"$P\,x$ holds $\lambda(dx)$-almost surely" $(P \hookrightarrow \Omega)$

for some $Q \hookrightarrow \Omega$, $P \supseteq Q$, $\Pr_\lambda Q^c = 0$

$\Leftrightarrow \Pr Q = 1$

<u>Example</u> $(\xi, \zeta \in \mathbb{H}^\Omega)$

$\xi = \zeta$ a.s. when $\Pr_{w \sim \lambda}[\xi w \neq \zeta w] = 0$

Integrating random variables

in Qbs!

$$(-)_+, (-)_- : \overline{\mathbb{R}}^\Omega \longrightarrow W^\Omega$$

$$\xi_+ := \max(\xi, 0) \qquad \xi_- := \max(-\xi, 0)$$

So: $\xi = \xi_+ - \xi_-$

$$\int : P\Omega \times W^\Omega \longrightarrow W$$

$$\int \lambda\, \xi := \int \lambda\, \xi_+ - \int \lambda\, \xi_-$$

$\int$ respects a.s. equality:

$\xi = \zeta$ (λ-a.s.)

$\Rightarrow \int \lambda\, \xi = \int \zeta$.

Example

$$\begin{tikzcd} \mathrm{ASConverge}(\overline{\mathbb{R}})^{\Omega} := \left\{ \vec{z} \in \overline{\mathbb{R}}^{\mathbb{N}\times\Omega} \,\middle|\, \Pr_{\omega\sim\lambda}\left[\lim z_n \omega \neq \bot\right] \right\} \arrow[d, hook] \\ \overline{\mathbb{R}}^{\mathbb{N}\times\Omega} \end{tikzcd}$$

So:

$\lim^{as} : \overline{\mathbb{R}}^{\mathbb{N}\times\Omega} \longrightarrow \overline{\mathbb{R}}^{\Omega}$ $\qquad \mathrm{Dom}\ \lim^{as} := \mathrm{ASConverge}(\overline{\mathbb{R}})^{\Omega}$

$$\lim^{as} \vec{z} := \lambda\omega.\ \limsup_{n\to\infty} f_n\, \omega$$

$\llcorner$ $\lim^{as}$ respects a.s. equality.

Thm (monotone convergence):

Let $\vec{\xi} \in \mathbb{W}^{\mathbb{N} \times \Omega}$ λ-a.s. monotone.

$$\xi = \lim_{n \to \infty}^{\text{as}} \xi_n \quad \text{(a.s.)}$$

$$\Downarrow$$

$$\int \lambda\, \xi = \lim_{n \to \infty} \int \lambda\, \xi_n$$

Lebesgue space

(Ω prob. space, $p \in [1, \infty)$)

$$\mathcal{L}^p_\Omega := \left\{ \xi \in \mathbb{R}^\Omega \,\middle|\, \int \lambda |\xi|^p < \infty \right\} \hookrightarrow \mathbb{R}^\Omega$$

Ensemble

$$\mathcal{L}_\Omega := \prod_{\substack{\lambda \in P\Omega \\ p \in [1,\infty)}} \mathcal{L}^p_{(\Omega,\lambda)} \hookrightarrow B_{\mathbb{R}^\Omega}^{P\Omega \times [1,\infty)}$$

$\llcorner$ $p \leq q \Rightarrow \mathcal{L}^p_\Omega \supseteq \mathcal{L}^q_\Omega$

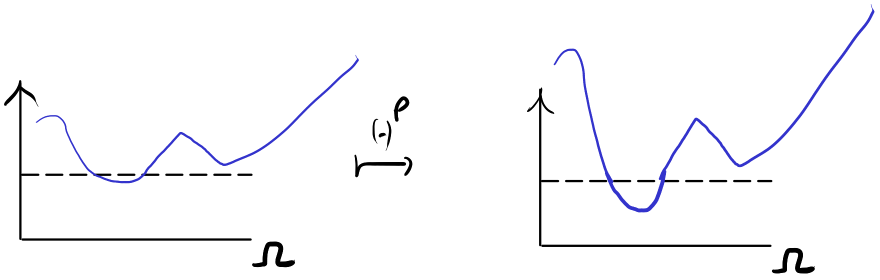
$\mathcal{L}^p$ semi norms

$$\|-\| : \coprod_{p,\lambda} \mathcal{L}^p_{(\Omega,\lambda)} \longrightarrow \mathbb{R}_{\geq 0} \qquad \|\xi\|_p := \sqrt[p]{\int \lambda |\xi|^p}$$

$\mathcal{L}^2$ inner product

$$\langle -, - \rangle : \coprod_{p,\lambda} \mathcal{L}^p_{(\Omega,\lambda)} \times \mathcal{L}^p_{(\Omega,\lambda)} \longrightarrow \mathbb{R}$$

$$\langle \xi, \zeta \rangle_{p,\lambda} := \int \lambda\, \xi \zeta$$



Statistics

Expectation

$\mathbb{E} : \coprod_\lambda \mathcal{L}^1 \longrightarrow \mathbb{R}$

$\mathbb{E}_\lambda \xi := \int \lambda \xi$

Covariance and Correlation

$\mathrm{Cov}, \mathrm{Corr} : \coprod_\lambda \mathcal{L}^2 \longrightarrow \mathbb{R}$

$\mathrm{Cov}(\xi, \zeta) := \langle \xi - \mathbb{E}\xi, \zeta - \mathbb{E}\zeta \rangle$

$\mathrm{Corr}(\xi, \zeta) := \dfrac{\langle \xi, \zeta \rangle}{\|\xi\|_2 \cdot \|\zeta\|_2} = \cos(\mathrm{angle}(\xi, \zeta))$

Sequential limits

$\text{Cauchy}\,\mathcal{L}^p_\Omega \hookrightarrow (\mathcal{L}^p)^{\mathbb{N}}$

$$\text{Cauchy}\,\mathcal{L}^p_\Omega := \left\{ \vec{\Sigma} \;\middle|\; \forall \varepsilon \in \mathbb{Q}^+ \, \exists k \in \mathbb{N} \, \forall m, n \geq k. \; \| \Sigma_{k+n} - \Sigma_{k+m} \|_p < \varepsilon \right\}$$

Thm: $\mathcal{L}^p_\Omega$ is Cauchy-Complete

$$\lim^p : \text{Cauchy}\,\mathcal{L}^p \longrightarrow \mathcal{L}^p$$

Why?

1. Every Cauchy sequence has an a.s. converging subseq.
2. We can find it measurably

Example

Thm (dominated convergence)

For $\vec{\xi}, \zeta \in \mathcal{L}^1$ s.t. $\xi_n \le \zeta$ a.s.:

1. $\lim^{as} \vec{\xi} \in \mathcal{L}^1$
2. $\lim^{1} \vec{\xi} = \lim^{as} \vec{\xi}$
3. $\lim_{n\to\infty} \int \lambda\, \xi_n = \int \lambda \lim_{n\to\infty} \xi_n$

Separability

Def: $\mathcal{L}^p$ separable: has countable dense subset

Fact: Separability is property of λ_Ω:

TFAE:

- $\exists p \geq 1.\ \mathcal{L}^p$ separable
- $\forall p \geq 1.\ \mathcal{L}^p$ separable

Measurable separability in $I \hookrightarrow P\Omega \times [1,\infty)$

$$\vec{\beta} : \prod_{(\lambda,p)\in I} {\mathcal{L}^p_{(\Omega,\lambda)}}^{\mathbb{N}} \quad \text{s.t.}$$

$$\{\vec{\beta}^{\lambda,p}_n \mid n \in \mathbb{N}\} \text{ dense in } \mathcal{L}^p_{(\Omega,\lambda)}$$

Prop. - every sbs S measurably separable in

$$PS \times [1,\infty)$$

- $I \hookrightarrow P\Omega \times \{2\}$ measurably separable

$$\Rightarrow \exists\, \vec{\beta} \in \prod_{\lambda \in I} \mathcal{L}^2_{(\Omega,\lambda)} \text{ orthonormal system}$$

$\langle \beta_n, \beta_m \rangle = 0$

$\|\beta_n\|_2 = 1$

(β_n) dense

Example

Let $S \hookrightarrow \mathcal{L}^2$ closed vector subspace.

orthogonal decomposition

linear in fact.

$$\langle P, P^{\perp}\rangle : \lfloor \mathcal{L}^P \rfloor \to \lfloor S \rfloor \times \lfloor S^{\perp} \rfloor$$

When S is separable with orthonormal system β

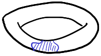
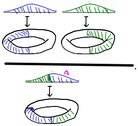
We have a measurable version of

$\langle P, P^{\perp}\rangle : \mathcal{L}^2 \to S \times S^{\perp}$

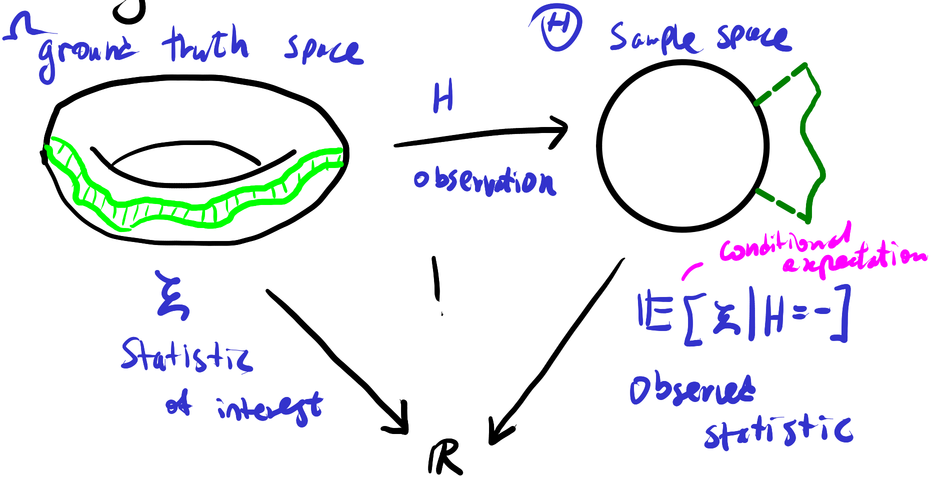
$$P\xi := \sum_{n=0}^{\infty} \langle \xi, \beta_n \rangle \beta_n \qquad P^{\perp} := \mathrm{Id} - P.$$

Agenda

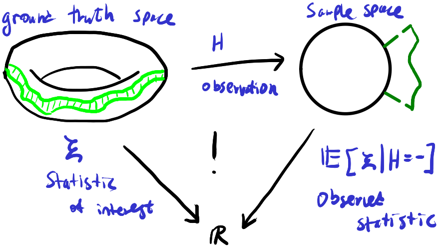
Slogan: Measurable by Type

- Borel sets 
- Qbs:
  def., constructions, 
  partiality, refinement $\rightharpoonup$, $\hookrightarrow$, $\coprod$, $\prod$
- Measures & integration δ, $\int$, $\fint$, D, $\otimes$, P
- Random variable spaces $\mathcal{L}^p$, $\|-\|_p$, $\mathbb{E}$
- Conditional expectation

Kolmogorov's Conditional Expectation



Kolmogorov's Conditional Expectation



A conditional expectation of $\xi \in \mathcal{L}^1_\Omega$ wrt $H : \Omega \to \Theta$ is $\zeta \in \mathcal{L}^1_\Theta$ s.t. for all $A \in \mathcal{B}_\Theta$:

$$\int_A \mu \zeta = \int_{H^{-1}[A]} \lambda \xi$$

where $\mu := H_* \lambda$

Conditional expectations

1. unique a.s.

2. fundamental to modern Probability, eg:

   a martingale

$$\begin{tikzcd}
\cdots & \Theta_n \arrow[l] \arrow[dr, "\xi_n"'] & & \Theta_{n+1} \arrow[ll, "H_n"'] \arrow[dl, "\xi_{n+1}"] & \cdots \arrow[l] \\
& & \mathbb{R} \arrow[u, phantom, "!"] & &
\end{tikzcd}$$

s.t. $\xi_n = \mathbb{E}[\xi_{n+1} \mid H_n = -]$

Thm (Existence)

- $\exists$ $\mathbb{E}[- \mid H = -] : \lfloor \mathcal{L}^1_{(\Omega,\lambda)} \rfloor \longrightarrow \mathcal{L}^1_{(\Theta,\mu)}$

- When (Ω, λ) is Separable

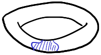
$\mathbb{E}[- \mid H = -] : \mathcal{L}^1_{(\Omega,\lambda)} \longrightarrow \mathcal{L}^1_{(\Theta,\mu)}$

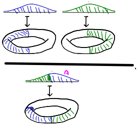
- When Θ is $\mathbb{I}$-measurably separable

$\mathbb{E}_{-}[- \mid - = -] : \prod_{\substack{H \in \Theta^{\Omega} \\ \lambda \in H^{-1}_{*}[\mathbb{I}]}} \mathcal{L}^1_{(\Omega,\lambda)} \longrightarrow \mathcal{L}^1_{(\Theta,\mu)}$

Agenda

Slogan: Measurable by Type

- Borel sets 
- Qbs:



  def., constructions, partiality, refinement $\rightharpoonup, \Rightarrow, \amalg, \prod$
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