A domain theory for statistical probabilistic programming

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Department of Computer Science and Technology
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Statistical probabilistic programming

$$[\![-]\!] : \text{programs} \to \text{unnormalised distributions}$$

- Bayesian inference: compiler computes normalisation
- Continuous types: $\mathbb{R}, [0, \infty]$
- Probabilistic effects:

normally distributed sample

$$\mathbf{sample}(\mu, \sigma) : \mathbb{R}$$

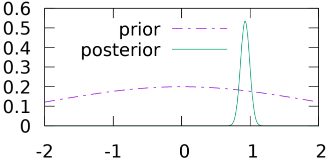
scale distribution by r

$$\frac{r : [0, \infty]}{\mathbf{score}(r) : \mathbf{1}}$$

conditioning/fitting to observed data with likelihood

prior

$$[\![\mathbf{sample}(0, 2)]\!]$$



$$\left[\!\!\left[\begin{array}{l}\mathbf{let}\ x{=}\mathbf{sample}(0{,}2)\\ \mathbf{in}\ \mathbf{score}(normalPdf(1.1|\ x, \tfrac{1}{4}));\\ \quad \mathbf{score}(normalPdf(1.9|2x, \tfrac{1}{4}));\\ \quad \mathbf{score}(normalPdf(2.7|3x, \tfrac{1}{4}));\\ \quad x\end{array}\right]\!\!\right]$$

posterior

Statistical probabilistic programming

Exact Bayesian inference using disintegration [Shan-Ramsey'17]

- Commutativity/exchangability/Fubini

$$\left[\!\!\left[\begin{array}{l}\mathbf{let}\ x = K\ \mathbf{in}\\ \mathbf{let}\ y = L\ \ \mathbf{in}\\ f(x,y)\end{array}\right]\!\!\right] = \left[\!\!\left[\begin{array}{l}\mathbf{let}\ y = L\ \ \mathbf{in}\\ \mathbf{let}\ x = K\ \mathbf{in}\\ f(x,y)\end{array}\right]\!\!\right] \qquad \begin{array}{c}\int [\![K]\!]\,(\mathrm{d}x) \int [\![L]\!]\,(\mathrm{d}y) f(x,y)\\ =\\ \int [\![L]\!]\,(\mathrm{d}y) \int [\![K]\!]\,(\mathrm{d}x) f(x,y)\end{array}$$

probabilitity distributions ✔

σ-finite distributions ✔

arbitrary distributions ✗

not closed under push-forward

s-finite distributions ✔

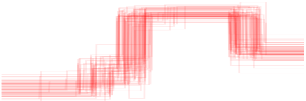
full definability [Staton'17]

Statistical probabilistic programming

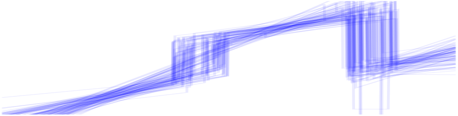
Express continuous distributions using:

- Higher-order functions:

(e.g. generative random function models)



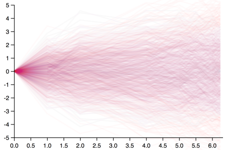
piecewise(random-constant)



piecewise(random-linear)

- Term recursion: (e.g. Gaussian random walk)

$$rw(x, \sigma) = \lambda(). \quad \text{// thunk}$$
$$\mathbf{let}\, y = \mathbf{sample}(x, \sigma)$$
$$\mathbf{in}\, (x, rw\,(y, \sigma))$$



- Type recursion (à la FPC) (e.g. dynamic types, IRs)

$$Dynamic = \mu\alpha.\{\mathrm{Val}(\mathbb{R}) \mid \mathrm{Fun}(\alpha \to \alpha)\}$$

Application: modular Bayesian inference

Resample-Move Sequential Monte Carlo

[Ścibior et al.'18a+b]

resamples particles moves

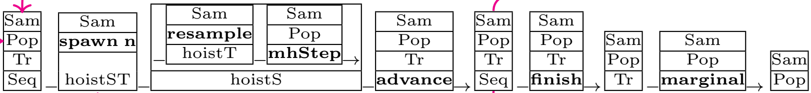
recursion

```
rmsmc k n t =
  marginal . finish . compose k (
    advance . hoistS (
      compose t mhStep . hoistT resample
    )
  ) . hoistST (spawn n >>)
```

inference representation

higher order

inference transformer



inference transformation (invariant preserving)

recursive types

ProbProg: Important Language Features

Church / WebPPL / Venture	sample	$\mathbb{R}$	score	higher order	term rec	type rec	Fubini (commute)
sets + probability	✓	✗	✗	✓	✗	✗	✓
meas space + subprobability	✓	✓	✗	✗	1^{st}	✗	✓
CPO + subprobability	✓	✓	✗	✓	✓	✓	?
cont domain + subprobability [Jones-Plotkin'89]	✓	✓	✗	✗	1^{st}	✗	✓
⋮ [Jung-Tix'98]	⋮	⋮	⋮	⋮	⋮	⋮	⋮
meas + s-finite distributions [Staton'17]	✓	✓	✓	✗	1^{st}	✗	✓
qbs + s-finite distributions [Heunen et al'17, Ścibior et al'18]	✓	✓	✓	✓	1^{st}	✗	✓
coh/meas cone + probability [Ehrhard-Pagani-Tasson'18, Ehrhard-Tasson'15-'19]	✓	✓ / ✗	✗	✓	✓	? / ✓	? / ✓
ωqbs + s-finite distributions [This work]	✓	✓	✓	✓	✓	✓	✓

Summary

Contribution

- $\omega\mathbf{Qbs}$: a category of pre-domain quasi-Borel spaces
- M: commutative probabilistic powerdomain over $\omega\mathbf{Qbs}$
- Axiomatic treatment of measure and domain theory in $\omega\mathbf{Qbs}$
- Adequacy: $(\omega\mathbf{Qbs}, M)$ adequately interprets:
  - Statistical FPC
  - Untyped Statistical λ-calculus

This talk

- $\omega\mathbf{Qbs}$
- A probabilistic powerdomain
- Axiomatic treatment
- Characterising $\omega\mathbf{Qbs}$

Iso-recursive types: FPC

[Fiore-Plotkin'94]

type variable contexts
$\Delta = \{\alpha_1, \ldots, \alpha_n\}$

$$\frac{\Delta, \alpha \vdash_{\mathrm{k}} \tau : \mathsf{type}}{\Delta \vdash_{\mathrm{k}} \mu\alpha.\tau : \mathsf{type}}$$

type recursion

$$\begin{aligned} Lam = \mu\alpha.\{&\mathrm{Bool}\{\mathrm{True} \mid \mathrm{False}\} \\ &\mid \mathrm{App}(\alpha * \alpha) \\ &\mid \mathrm{Abs}(\alpha \to \alpha)\} \end{aligned}$$

$\tau = \mu\alpha.\sigma$

$$\frac{\Gamma \vdash t : \sigma[\alpha \mapsto \tau]}{\Gamma \vdash \tau.\mathbf{roll}\,(t) : \tau} \qquad \frac{\Gamma \vdash t : \tau \qquad \Gamma, x : \sigma[\alpha \mapsto \tau] \vdash s : \rho}{\Gamma \vdash \mathbf{match}\, t\, \mathbf{with}\, \mathbf{roll}\, x \Rightarrow s : \rho}$$

Iso-recursive types: FPC

[Fiore-Plotkin'94]

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$\omega\mathbf{Cpo}$-enriched category of domains

$$[\![\Delta \vdash_{\mathrm{k}} \tau : \mathsf{type}]\!] : (\mathcal{C}^{\mathrm{op}})^n \times \mathcal{C}^n \to \mathcal{C}$$

locally continuous functor

$$[\![\Delta \vdash_{\mathrm{k}} \mu\alpha.\tau : \mathsf{type}]\!] = \text{minimal invariants}$$

[Freyd'91,92, Pitts'96]

Challenge

- probabilistic powerdomain
- commutativity/Fubini
- domain theory
- higher-order functions

continuous domains
[Jones-Plotkin'89]

open problem
[Jung-Tix'98]

traditional approach:

domain $\mapsto$ Scott-open sets $\mapsto$ Borel sets $\mapsto$ distributions/valuations

our approach:

as in
[Ehrhard-Pagani-Tasson'18]

$$(\text{domain}, \text{quasi-Borel space}) \mapsto \text{distributions}$$

separate
but compatible

Rudimentary measure theory

Borel sets

- $[a, b]$ Borel
- A Borel $\implies A^{\complement}$ Borel
- $(A_n)_{n\in\mathbb{N}}$ Borel $\implies$
  $\bigcup_{n\in\mathbb{N}} A_n$ Borel

Measurable functions $f : \mathbb{R} \to \mathbb{R}$

$f^{-1}[A]$ Borel $\impliedby$ A Borel

Measures μ : Borel $\to [0, \infty]$

- monotone:
  $A \subseteq B \implies \mu(A) \leq \mu(B)$
- Scott-continuous:
  $A_0 \subseteq A_1 \subseteq \ldots \implies \mu\left(\bigcup_n A_n\right) = \bigvee_n \mu(A_n)$
- strict ($\mu\emptyset = 0$) and additive ($\mu(A \uplus B) = \mu A + \mu B$)

Rudimentary measure theory

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1 dimensional

Example (Lebesgue measures)

$\lambda[a, b] = b - a$ on $\mathbb{R}$

$(\lambda \otimes \lambda)\left([a, b] \times [c, d]\right) =$
$(b - a)(d - c)$ on $\mathbb{R}^2$

2 dimensional

Example (Push-forward measure)

$f_*\mu(A) := \mu\left(f^{-1}[A]\right)$

Borel set

measure

$f : \mathbb{R} \to \mathbb{R}$

Quasi-Borel pre-domains

ω-qbs:

$$X = (\underbrace{\underline{X}}_{\text{set}}, \underbrace{\leq_X}_{\text{partial order on } \underline{X}}, \underbrace{M_X}_{\text{subset of functions } \mathbb{R} \to \underline{X} \text{ “random elements”}})$$

Quasi-Borel pre-domains

$\boldsymbol{\omega}$**-qbs**:

set

partial order on $\underline{X}$

subset of functions
$\mathbb{R} \to \underline{X}$
"random elements"

$$X = (\underline{X}, \leq_X, M_X)$$

ω-cpo

- $x_0 \leq x_1 \leq x_2 \leq \ldots \qquad \Longrightarrow \qquad \exists \bigvee_n x_n$

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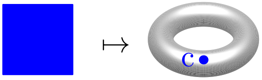
ω-cpo

quasi-Borel
space

- $\lambda\_.x \in M_X$



Quasi-Borel pre-domains

set partial order on $\underline{X}$ subset of functions $\mathbb{R} \to \underline{X}$ "random elements"

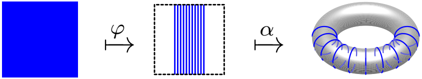
ω-qbs:

$$X = (\underline{X}, \leq_X, M_X)$$

ω-cpo quasi-Borel space

$\mathbb{R} \xrightarrow{\varphi}_{\text{Borel}} \mathbb{R}$

- $\lambda\_.x \in M_X$
- $\alpha \in M_X \implies \alpha \circ \varphi \in M_X$



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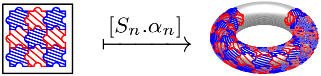
- $(\alpha_n \in M_X)_{n \in \mathbb{N}} \implies [r \in S_n.\alpha(r)] \in M_X$



Borel measurable
countable partition
$\mathbb{R} = \biguplus_{n \in \mathbb{N}} S_n$

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$\mathbb{R} = \biguplus_{n \in \mathbb{N}} S_n$

s.t.:

pointwise
ω-chain

pointwise
lub

$$(\alpha_n) \in M_X^\omega \implies \bigvee_n \alpha_n \in M_X$$

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pointwise ω-chain

Borel measurable countable partition $\mathbb{R} = \biguplus_{n \in \mathbb{N}} S_n$

pointwise lub

$$(\alpha_n) \in M_X^\omega \implies \bigvee_n \alpha_n \in M_X$$

Morphisms $f : X \to Y$: Scott continuous qbs maps

monotone and $f \bigvee_n x_n = \bigvee_n f\, x_n$

$\forall \alpha \in M_X.$ $f \circ \alpha \in M_Y$

Quasi-Borel pre-domains

$$\begin{tikzcd} \omega\mathbf{Qbs} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathbf{Qbs} \arrow[l, bend left] \end{tikzcd}$$

Example

$S = (\underline{S}, \Sigma_S)$ measurable space

$$(\underline{S}, =, \{\alpha : \mathbb{R} \to \underline{S} | \alpha \text{ Borel measurable}\})$$

so $\mathbb{R} \in \omega\mathbf{Qbs}$

Reminder

wqbs: $X = (\underline{X}, \leq_X, M_X)$

- $\lambda\_.x \in M_X$
- $\alpha \in M_X \implies \alpha \circ \varphi \in M_X$
- $(\alpha_n \in M_X)_{n \in \mathbb{N}} \implies [r \in S_n.\alpha(r)] \in M_X$

s.t.:

$$(\alpha_n) \in M_X^\omega \qquad \implies \qquad \bigvee_n \alpha_n \in M_X$$

Quasi-Borel pre-domains

Example

lubs of step functions

$P = (\underline{P}, \leq_P)$ ω-cpo

$$\omega\mathbf{Qbs} \underset{}{\overset{}{\rightleftarrows}} \omega\mathbf{Cpo} \quad \top$$

$$\left(\underline{P}, \leq_P, \left\{ \bigvee_k [\_ \in S_n^k . a_n^k] \middle| \forall k . \mathbb{R} = \biguplus_n S_n^k \right\}\right)$$

so $\mathbb{L} = ([0,\infty], \leq, \{\alpha : \mathbb{R} \to [0,\infty] | \alpha$ Borel measurable$\}) \in \omega\mathbf{Qbs}$

Reminder

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$$(\alpha_n) \in M_X^\omega \qquad \implies \qquad \bigvee_n \alpha_n \in M_X$$

Quasi-Borel pre-domains

$$\omega\mathbf{Qbs} \circlearrowright$$

Example

X ω-qbs

$$X_\perp := \left(\{\perp\} + \underline{X}, \perp \leq \underline{X}, \left\{[S.\perp, S^{\complement}.\alpha] \middle| \alpha \in M_X, S \text{ Borel}\right\}\right)$$

Reminder

wqbs: $X = (\underline{X}, \leq_X, M_X)$

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$$(\alpha_n) \in M_X^\omega \qquad \implies \qquad \bigvee_n \alpha_n \in M_X$$

Quasi-Borel pre-domains

Products

$$\underline{X_1 \times X_2} = \underline{X}_1 \times \underline{X}_2 \qquad x \leq y \iff \forall i. x_i \leq y_i$$

$$M_{X_1 \times X_2} = \{(\alpha_1, \alpha_2) : \mathbb{R} \to \underline{X}_1 \times \underline{X}_2 | \forall i. \alpha_i \in M_{X_i}\}$$

correlated
random elements

Quasi-Borel pre-domains

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Theorem

$\omega\mathbf{Qbs} \to \omega\mathbf{Cpo} \times \mathbf{Qbs}$ *creates limits*

Quasi-Borel pre-domains

Products

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Exponentials

- $\underline{Y^X} = \{f : \underline{X} \to \underline{Y} | f$ Scott continuous qbs morphism$\}$
  $= \mathbf{Qbs}(X, Y)$
- $f \le g \iff \forall x \in \underline{X}. f(x) \le g(x)$
- $M_{Y^X} = \left\{ \alpha : \mathbb{R} \to \underline{Y^X} \middle| \begin{array}{l} \text{uncurry}\, \alpha : \mathbb{R} \times X \to Y \\ \text{Scott continuous qbs morphism} \end{array} \right\}$

so $\underline{Y}^{\mathbb{R}} = M_Y$

Fundamentals of measure theory

s-finite measures

- μ_n **bounded**: $\mu_n(\mathbb{R}) < \infty$
- μ **s-finite**: $\mu = \sum_n \mu_n$, μ_n bounded

Randomisation Theorem

Every s-finite measure is a push-forward of Lebesgue:

$$\mu \text{ s-finite} \implies \mu = f_* \lambda \text{ for some } f : \mathbb{R} \to \mathbb{R}_\perp$$

Transfer principle

$\tau_* \lambda = \lambda \otimes \lambda$ for some measurable $\tau : \mathbb{R} \to (\mathbb{R} \times \mathbb{R})_\perp$

Randomisation monad structure

- $(X_\perp)^{\mathbb{R}}$
- $\mathrm{return}_X(x) : r \in [0,1] \mapsto x$
- $(\alpha \gg\!= f) : \mathbb{R} \xrightarrow{\tau} \mathbb{R} \times \mathbb{R} \xrightarrow{\mathbb{R}\times\alpha} \mathbb{R} \times X \xrightarrow{\mathbb{R}\times f} \mathbb{R} \times (Y_\perp)^{\mathbb{R}} \xrightarrow{\mathrm{eval}} Y$

$\mathbb{R} \to X_\perp$ $\quad$ $X \to (Y_\perp)^{\mathbb{R}}$

- sample from randomisation of normal distribution
- $\mathrm{score}(r) : r' \in [0, |r|] \mapsto ()$

monad laws fail
(associativity)

Lebesgue integration

$$\begin{array}{c}\mathbb{R}\\ \alpha\downarrow\\ X_\perp\end{array},\ \begin{array}{c}X\\ f\downarrow\\ \mathbb{L}\end{array} \mapsto \quad \begin{array}{c}\mathrm{Dom}(\alpha)\\ f\circ\alpha\downarrow_{\text{Borel}}\\ \mathbb{L}\end{array} \quad \mapsto \int_{\mathrm{Dom}(\alpha)} f\circ\alpha(x)\lambda(\mathrm{d}x)$$

Lebesgue measure

$\mathrm{Dom}\,(\alpha)$ Borel

Borel maps and natural order

$$\int : (X_\perp)^{\mathbb{R}} \longrightarrow \mathbb{L}^{\mathbb{L}^X}$$

preserves monad structure

randomisation monad structure

continuation monad

A probabilistic powerdomain

$$\begin{tikzcd}
(X_\perp)^{\mathbb{R}} \arrow[rr, "\int"] \arrow[dr, two heads] & {} \arrow[d, phantom, "="] & \mathbb{L}^{\mathbb{L}^X} \\
& MX \arrow[ur, tail] &
\end{tikzcd}$$

MX: randomisable integration operators

A probabilistic powerdomain

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, two heads] & {} \arrow[d, phantom, "="] & Y \\
& \left(\mathrm{Cl}_\omega f[X], \leq_Y, \mathrm{Cl}_\omega^{Y^{\mathbb{R}}} f \circ [M_X]\right) \arrow[ur, tail] &
\end{tikzcd}
$$

lubs of lubs of ...
f images

pointwise order

$(\mathcal{E}, \mathcal{M}) :=$ (densely strong epi, full mono) factorisation system

A probabilistic powerdomain

$\mathcal{E} =$ densely strong epis closed under:

- products: $e_1, e_2 \in \mathcal{E} \implies e_1 \times e_2 \in \mathcal{E}$
- lifting: $e \in \mathcal{E} \implies e_\perp \in \mathcal{E}$
- random elements: $e \in \mathcal{E} \implies e^{\mathbb{R}} \in \mathcal{E}$

$\implies$ M strong monad for sampling + conditioning

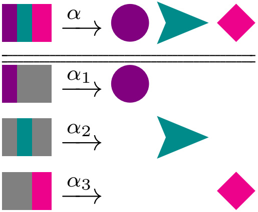
[Kammar-McDermott'18]

A probabilistic powerdomain

$$\begin{tikzcd}
(X_\perp)^{\mathbb{R}} \arrow[rr] \arrow[dr, two heads] \arrow[rr, phantom, shift right=4ex, "="] & & \mathbb{L}^{\mathbb{L}^X} \\
& MX \arrow[ur, tail] &
\end{tikzcd}$$

- M locally continuous $\implies$ may appear in domain equations
- M commutative $\implies$ satisfies Fubini
- M models synthetic measure theory
  $M \sum_{n\in\mathbb{N}} X_n \cong \prod_{n\in\mathbb{N}} M X_n$

[Kock'12,
Ścibior et al.'18]



Valuations

Proposition

For standard Borel X:

$$MX \cong \left\{ \mu\big|_{\textit{Scott opens}} \,\middle|\, \mu \textit{ is s-finite} \right\}$$

Theorem (joint with Alex Simpson)

Quasi-Borel domains freely extend continuous domains:

$$\begin{tikzcd}
 & \omega\mathbf{Cpo} \arrow[dr] & \\
\mathbf{Cont}_\perp \arrow[ur, hook] \arrow[dr] \arrow[rr, phantom, "="] & & \omega\mathbf{Qbs} \arrow[dl] \\
 & \omega\mathbf{Cpo} \times \mathbf{Meas} &
\end{tikzcd}
\qquad
\begin{tikzcd}
(D, \leq) \arrow[rr, mapsto, bend left] \arrow[dr, mapsto] & & \left(D, \leq, \mathrm{Cl}_\omega \, \substack{\textbf{step}\\\textbf{functions}}\right) \arrow[dl, mapsto] \\
 & \left((D, \leq), (D, \sigma(\substack{\textbf{Scott}\\\textbf{opens}}))\right) &
\end{tikzcd}$$

Axiomatic domain theory

[Fiore-Plotkin'94, Fiore'96]

Structure

- Total map category: $\omega\mathbf{Qbs}$
- Admissible monos: **Borel-open** map $m : X \rightarrowtail Y$:

strong mono

qbses

$$\forall \beta \in M_Y . \qquad \beta^{-1}[m[X]] \in \mathcal{B}(\mathbb{R})$$

  take Borel-Scott open maps as admissible monos
- $\mathbf{Pos}$-enrichment: pointwise order
- Pointed monad on total maps: the powerdomain

$\Longrightarrow$ model axiomatic domain theory

$\Longrightarrow$ solve recursive domain equations

Axiomatic domain theory

Structure

$\mathfrak{D}$	total map category $\omega\mathbf{Qbs}$
$f \leq g$	$\mathbf{Pos}$-enrichment pointwise order
$\mathcal{M}_{\mathfrak{D}}$	admissible monos Borel-Scott opens
T	monad for effects power-domain
m	partiality encoding $m : -_{\perp} \to T,\ \perp \mapsto \underline{0}$

Axioms

$(\dashv)$	every object has a partial map classifier $\downarrow_X : X \to X_{\perp}$
(fup)	every admissible mono is full and upper-closed
$(\dashv_{\leq})$	$\lfloor - \rfloor$ is locally monotone
$(\bigvee)$	$\mathfrak{D}$ is $\omega\mathbf{Cpo}$-enriched
(U)	ω-colimits behave uniformly
$(\mathbb{1})$	$\mathfrak{D}$ has a terminal object
$(\to_{\leq})$	$\mathfrak{D}$ has locally monotone exponentials
$(+)$	locally continuous total coproducts
$(?!)$	$\mathbb{0} \to \mathbb{1}$ is admissible
$(\times_{\bigvee})$	$\mathfrak{D}$ has a locally continuous products
(CL)	$\mathfrak{D}$ is cocomplete
$(T_{\bigvee})$	T is locally continuous

Derived axioms/structure

$\mathbf{p}\mathfrak{D}$	partial map category
$-_{\perp}$	partiality monad
$(\dashv_{\bigvee})$	the adjunction $J \dashv L$ is locally continuous
$(\mathbf{p}_{\bigvee})$	$\mathbf{p}\mathfrak{D}$ is $\omega\mathbf{Cpo}$-enriched
$(\mathbb{1}_{\leq})$	$\mathbf{p}\mathfrak{D}$ has a partial terminal
$(\otimes)$	$\mathbf{p}\mathfrak{D}$ has partial products
$(\otimes_{\bigvee})$	$(\otimes)$ is locally continuous
$(\to_{\bigvee})$	$\mathfrak{D}$ has locally continuous exponentials
$(\rightharpoonup_{\bigvee})$	$\mathbf{p}\mathfrak{D}$ has locally continuous partial exponentials
$(\mathbf{p}CL)$	$\mathbf{p}\mathfrak{D}$ is cocomplete
$(\mathbf{p}{+}_{\bigvee})$	$\mathbf{p}\mathfrak{D}$ has locally continuous partial coproducts
(BC)	$J : \mathfrak{D} \hookrightarrow \mathbf{p}\mathfrak{D}$ is a bilimit compact expansion

Characterising $\omega\mathbf{Qbs}$

$$\begin{tikzcd}
\mathbf{Sbs} \arrow[rr, hook, "\text{Yoneda}"] \arrow[dr, hook, "\text{Yoneda}"'] & {} \arrow[d, phantom, "="] & {[\mathbf{Sbs}^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{cpp}}} \\
& \mathsf{SepSh} \arrow[ur, hook] &
\end{tikzcd}$$

countable
product preserving
[Staton et al.'16]

$F : \mathbf{Sbs}^{\mathrm{op}} \to \mathbf{Set}$ separated: $F\mathbb{R} \xrightarrow{\left(F(\mathbb{R} \xleftarrow{r} \mathbb{1})\right)_{r \in \mathbb{R}}} (F\mathbb{1})^{\mathbb{R}}$ injective

Thm: $\mathbf{Qbs} \simeq$ SepSh

[Heunen et al.'17]

$$\begin{tikzcd}
\mathbf{Sbs} \arrow[rr, hook, "\text{Yoneda}"] \arrow[dr, hook, "\text{Yoneda}"'] & {} \arrow[d, phantom, "="] & {[\mathbf{Sbs}^{\mathrm{op}}, \omega\mathbf{Cpo}]_{\mathrm{cpp}}} \\
& \omega\mathsf{SepSh} \arrow[ur, hook] &
\end{tikzcd}$$

$f(x) \leq f(y)$
$\implies x \leq y$

$F : \mathbf{Sbs}^{\mathrm{op}} \to \omega\mathbf{Cpo}$ ω-separated: $F\mathbb{R} \xrightarrow{\left(F(\mathbb{R} \xleftarrow{r} \mathbb{1})\right)_{r \in \mathbb{R}}} (F\mathbb{1})^{\mathbb{R}}$ full

Thm: $\omega\mathbf{Qbs} \simeq_{\omega\mathbf{Cpo}} \omega$SepSh

Characterising $\omega\mathbf{Qbs}$

Grothendieck quasi-topos $\mathbf{Qbs}$
strong subobject classifier:

$$\underline{\Omega} = 2 \qquad\qquad M_\Omega = 2^{\mathbb{R}}$$

strong monos:

$X \overset{f}{\rightarrowtail} Y$

$(f\circ)^{-1}[M_Y] = M_X$

$P^2 \xrightarrow{\leq_P} \Omega$

qbs

ω-chain$(P) \xrightarrow{\bigvee} P$

Internal ω-cpo P: $(\underline{P}, \leq_P, \bigvee)$

$+$ internal quasi-topos logic ω-cpo axioms

Theorem

$$\omega\mathbf{Qbs} \simeq \omega\mathbf{Cpo}(\mathbf{Qbs})$$

Characterising $\omega\mathbf{Qbs}$

By local presentability:

$$\omega\mathbf{Cpo} \simeq \mathrm{Mod}(\boldsymbol{\omega}\mathbf{cpo}, \mathbf{Set}) \qquad \mathbf{Qbs} \simeq \mathrm{Mod}(\mathbf{qbs}, \mathbf{Set})$$

essentially algebraic theories

$\boldsymbol{\omega}\mathbf{qbs}$:

$$\boldsymbol{\omega}\mathbf{cpo} \cup \mathbf{qbs} \cup \text{compatibility axiom}$$

Theorem

$$\omega\mathbf{Qbs} \simeq \mathrm{Mod}(\boldsymbol{\omega}\mathbf{qbs}, \mathbf{Set})$$

so $\omega\mathbf{Qbs}$ locally presentable, hence cocomplete

Summary

Contribution

- $\omega\mathbf{Qbs}$: a category of pre-domain quasi-Borel spaces
- M: commutative probabilistic powerdomain over $\omega\mathbf{Qbs}$
- Axiomatic treatment of measure and domain theory in $\omega\mathbf{Qbs}$
- Adequacy: $(\omega\mathbf{Qbs}, M)$ adequately interprets:
  - Statistical FPC
  - Untyped Statistical λ-calculus

[Fiore-Plotkin'94, Fiore'96]

This talk

- $\omega\mathbf{Qbs}$
- A probabilistic powerdomain
- Axiomatic treatment
- Characterising $\omega\mathbf{Qbs}$

Also in the paper

- Axiomatic domain theory
- Operational semantics à la [Borgström et al.'16]

ProbProg: Important Language Features

Church WebPPL Venture	sample	$\mathbb{R}$	score	higher order	term rec	type rec	Fubini (commute)
sets + probability	✓	✗	✗	✓	✗	✗	✓
meas space + subprobability	✓	✓	✗	✗	1^{st}	✗	✓
CPO + subprobability	✓	✓	✗	✓	✓	✓	?
cont domain + subprobability [Jones-Plotkin'89]	✓	✓	✗	✗	1^{st}	✗	✓
⋮ [Jung-Tix'98]	⋮	⋮	⋮	⋮	⋮	⋮	⋮
meas + s-finite distributions [Staton'17]	✓	✓	✓	✗	1^{st}	✗	✓
qbs + s-finite distributions [Heunen et al'17, Ścibior et al'18]	✓	✓	✓	✓	1^{st}	✗	✓
coh/meas cone + probability [Ehrhard-Pagani-Tasson'18, Ehrhard-Tasson'15-'19]	✓	✓ ✗	✗	✓	✓	? ✓	? ✓
ω**qbs + s-finite distributions** [This work]	✓	✓	✓	✓	✓	✓	✓

Presenting Pos

[cf. Adámek-Rosicky'94
Examples 3.35(1),(4)]

Sorts/arities

elem $\qquad$ ineq

Operations

$\mathrm{lower} : \mathrm{ineq} \to \mathrm{elem}$ $\qquad$ $\mathrm{upper} : \mathrm{ineq} \to \mathrm{elem}$ $\qquad$ $\mathrm{refl} : \mathrm{elem} \to \mathrm{ineq}$

$\mathrm{irrel} : \mathrm{ineq} \times \mathrm{ineq} \rightharpoonup \mathrm{ineq}$

$\mathrm{Def}(\mathrm{irrel}(e_1, e_2))$:
$\mathrm{lower}\,(e_1) = \mathrm{lower}\,(e_2)$
$\mathrm{upper}(e_1) = \mathrm{upper}(e_2)$

$\mathrm{antisym} : \mathrm{ineq} \times \mathrm{ineq} \rightharpoonup \mathrm{elem}$

$\mathrm{Def}(\mathrm{antisym}(e, e^{\mathrm{op}}))$:
$\mathrm{lower}\,(e) = \mathrm{upper}(e^{\mathrm{op}})$
$\mathrm{upper}(e) = \mathrm{lower}\,(e^{\mathrm{op}})$

$\mathrm{trans} : \mathrm{ineq} \times \mathrm{ineq} \rightharpoonup \mathrm{ineq}$

$\mathrm{Def}(\mathrm{trans}(e_1, e_2))$:
$\mathrm{upper}(e_1) = \mathrm{lower}(e_2)$

Axioms

$e_1 = \mathrm{irrel}(e_1, e_2) = e_2$ $\qquad$ $\mathrm{lower}(\mathrm{refl}(x)) = x = \mathrm{upper}(\mathrm{refl}(x))$

$\mathrm{lower}(e_1) = \mathrm{antisym}(e_1, e_2) = \mathrm{lower}(e_2)$

$\mathrm{lower}(\mathrm{trans}(e_1, e_2)) = \mathrm{lower}(e_1)$ $\qquad$ $\mathrm{upper}(\mathrm{trans}(e_1, e_2)) = \mathrm{upper}(e_2)$

Presenting $\omega\mathbf{Cpo}$

Add to $\mathbf{pos}$:

Operations

$\bigvee : \prod_{n\in\mathbb{N}} \mathrm{ineq} \rightharpoonup \mathrm{elem}$

Def($\bigvee_{n\in\mathbb{N}} e_n$):
$\mathrm{upper}(e_n) = \mathrm{lower}(e_{n+1})$
for each $n \in \mathbb{N}$

$\mathrm{ub}_k : \prod_{n\in\mathbb{N}} \mathrm{ineq} \rightharpoonup \mathrm{ineq}$

Def($\mathrm{ub}_k\,(e_n)_{n\in\mathbb{N}}$):
$\mathrm{upper}(e_n) = \mathrm{lower}(e_{n+1})$
for each $n \in \mathbb{N}$

$$\mathrm{lst} : \mathrm{elem} \times \prod_{n\in\mathbb{N}} \mathrm{ineq} \times \prod_{n\in\mathbb{N}} \mathrm{ineq} \rightharpoonup \mathrm{ineq}$$

Def($\mathrm{lst}(x, (e_n), (b_n))$):
$\mathrm{upper}(e_n) = \mathrm{lower}(e_{n+1})$
$\mathrm{upper}(b_n) = x$
$\mathrm{lower}\ (e_n) = \mathrm{lower}(b_n)$
for each $n \in \mathbb{N}$

Axioms

$\mathrm{lower}(\mathrm{ub}_k\,(e_n)) = \mathrm{lower}(e_k)$
$\mathrm{upper}(\mathrm{ub}_k\,(e_n)) = \bigvee (e_n)$

$\mathrm{lower}(\mathrm{lst}(x, (e_n), (b_n))) = \bigvee (e_n)$ $\quad$ $\mathrm{upper}(\mathrm{lst}(x, (e_n), (b_n)_n)) = x$

Presenting Qbs

Sorts/arities

$$\mathrm{elem} \qquad\qquad \mathrm{rand}$$

Operations

$\mathrm{ev}_r : \mathrm{rand} \to \mathrm{elem}$

$\mathrm{const} : \mathrm{elem} \to \mathrm{rand}$

$\mathrm{rearrange}_\varphi : \mathrm{rand} \to \mathrm{rand}$

$\mathrm{match}_{(S_i)_{i\in I}} : \prod_{i \in I} \mathrm{rand} \to \mathrm{rand}$

$\mathrm{ext} : \mathrm{rand} \times \mathrm{rand} \rightharpoonup \mathrm{rand}$

$\mathrm{Def}(\mathrm{ext}(\alpha, \beta))$:
$\mathrm{ev}_r(\alpha) = \mathrm{ev}_r(\beta)$
for each $r \in \mathbb{R}$

Axioms

$$\alpha = \mathrm{ext}(\alpha, \beta) = \beta \qquad\qquad \mathrm{ev}_r(\mathrm{const}\,(x)) = x$$

$$\mathrm{ev}_r(\mathrm{rearrange}_\varphi\, \alpha) = \mathrm{ev}_{\varphi(r)}\, \alpha$$

$$\mathrm{ev}_r \left(\mathrm{match}_{(S_i)_{i\in I}}\, (\alpha_i)_{i\in I}\right) = \mathrm{ev}_r(\alpha_i)$$

Presenting ωQbs

Sorts/arities

$$\text{elem} \qquad \text{ineq} \qquad \text{rand}$$

Operations

Add to $\boldsymbol{\omega}\mathbf{cpo}$ and $\mathbf{qbs}$:

$\bigsqcup : \prod_{n\in\mathbb{N}} \text{rand} \times \prod_{n\in\mathbb{N}, r\in\mathbb{R}} \text{ineq} \to \text{rand}$

$\text{Def} \bigsqcup((\alpha_n)_{n\in\mathbb{N}}, (e_n^r)_{n\in\mathbb{N}, r\in\mathbb{R}}) :$

$$\text{lower}(e_n^r) = \text{ev}_r(\alpha_n) \quad \text{upper}(e_n^r) = \text{ev}_r(\alpha_{n+1})$$

for each $n \in \mathbb{N}$, $r \in \mathbb{R}$

Axioms

Add:

$$\text{ev}_r\left(\bigsqcup\left((\alpha_n)_{n\in\mathbb{N}}, (e_n^r)_{n\in\mathbb{N}, r\in\mathbb{R}}\right)\right) = \bigvee (e_n^r)_{n\in\mathbb{N}}$$