A monad for full ground reference cells

Ohad Kammar, Paul B. Levy, Sean K. Moss, and Sam Staton
http://arxiv.org/abs/1702.04908

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Full ground references

ground reference cells

Dynamic allocation on the heap:

$\boxed{\mathbf{true}}$

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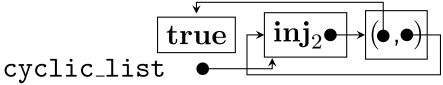


i.e.:

- linked lists;
- trees;
- graphs; etc.

Kripke semantics for local state

Semantics for full ground storage

- Sets-with-structure and structure preserving functions
- Monad over a bi-CCC

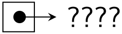
Extending the line of work of:

- Reynolds and Oles'82
- Moggi'90
- O'Hearn and Tennent'92,
- Stark'94
- Ghica'97
- Plotkin-Power'02
- Levy'02

Variance

Kripke semantics is functorial in the shape of the heap.
But the collection of heaps is not functorial:

$$\in \mathbf{H}\emptyset \qquad \in \mathbf{H}\{\ell_1 : \mathtt{ptr}\} \qquad \in \mathbf{H}\{\ell_1, \ell_2 : \mathtt{ptr}\}$$



Contribution

- Heaps functor over **initialisations**.
- A hiding/encapsulation monad over initialisations.
- A full ground references monad.
- Its decomposition into a global state transformation on the hiding monad.
- Evaluation:
  - An effect masking property.
  - Adequate semantics for a call-by-value calculus for full ground references.
  - Validation of the local state equations.
    [Plotkin-Power'02, Staton'10, Melliès'14]

$$\begin{tikzcd} \mathbf{Set}^{\mathbb{W}} \arrow[rr, bend right, "- \odot \mathbb{H}"'] \arrow[rr, phantom, "\top"] & & \mathbf{Set}^{\mathbb{E}} \arrow[ll, bend right, "\mathbb{H} \multimap -"'] \arrow[loop right, "P"] \end{tikzcd}$$

Full ground types

Full ground signature $\langle \mathbf{S}, \mathit{ctype} \rangle$

- Countably many $c \in \mathbf{S}$ **cell sorts**

which determine the **full ground types** $\gamma \in \mathbf{G}$ are:

$$\gamma ::= \mathbf{0} \mid \gamma_1 + \gamma_2 \mid \mathbf{1} \mid \gamma_1 * \gamma_2 \mid \mathbf{ref}_c$$

- A **content type** function $\mathit{ctype} : \mathbf{S} \to \mathbf{G}$

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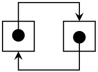
Example

$\mathbf{S} := \{\mathtt{ptr}\}$

$$\mathit{ctype}\ \mathtt{ptr} = \mathbf{ref}_{\mathtt{ptr}}$$



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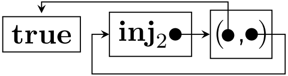
Example

$$\mathbf{S} := \{\mathtt{ptr}, \mathtt{data}, \mathtt{linked\_list}, \mathtt{list\_cell}\}$$

$$\begin{aligned}
\mathit{ctype}\ \mathtt{ptr} &= \mathbf{ref}_{\mathtt{ptr}} \\
\mathit{ctype}\ \mathtt{data} &= \mathbf{bool} \\
\mathit{ctype}\ \mathtt{linked_list} &= \mathbf{1} + \mathbf{ref}_{\mathtt{list_cell}} \\
\mathit{ctype}\ \mathtt{list_cell} &= \mathbf{ref}_{\mathtt{data}} * \mathbf{ref}_{\mathtt{linked_list}}
\end{aligned}$$



Worlds and injections

The category $\mathbb{W}$

Objects: Heap layouts/worlds

$$w = \{\ell_1 : c_1, \ldots, \ell_n : c_n\}.$$

Morphisms $\rho : w \to w'$: label preserving injections $\rho : w \rightarrowtail w'$

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Interpreting full ground types

Set $\mathbf{W} := [\mathbb{W}, \mathbf{Set}]$ and define:

$$[\![\mathbf{0}]\!] \coloneqq \mathbb{0} \qquad [\![\gamma_1 + \gamma_2]\!] \coloneqq [\![\gamma_1]\!] + [\![\gamma_2]\!]$$

$$[\![\mathbf{1}]\!] \coloneqq \mathbb{1} \qquad [\![\gamma_1 + \gamma_2]\!] \coloneqq [\![\gamma_1]\!] \times [\![\gamma_2]\!]$$

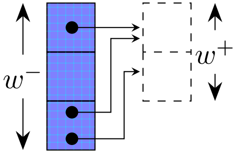
$$[\![\mathbf{ref}_c]\!]\, w \coloneqq \{\ell \in w | w(\ell) = c\} \qquad [\![\mathbf{ref}_c]\!]\, \rho(\ell) \coloneqq \rho(\ell)$$

Heaps

Heaplets

Define $\underline{\mathbb{H}} : \mathbb{W}^{\mathrm{op}} \times \mathbb{W} \to \mathbf{Set}$:

$$\underline{\mathbb{H}}(w^-, w^+) \coloneqq \prod_{(\ell:c)\in w^-} [\![\mathit{ctype}\ c]\!]\, w^+$$



Heaps

$$\mathbb{H}w \coloneqq \underline{\mathbb{H}}(w, w)$$

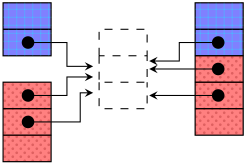
Combining heaplets

Independent coproducts [Simpson MFPS'17]

We have morphisms: $w_1 \xrightarrow{\iota_1^{\oplus}} w_1 \oplus w_2 \xleftarrow{\iota_2^{\oplus}} w_2$ in $\mathbb{W}$

Canonical isomorphisms

- $\mathbb{H}^{\oplus} : \underline{\mathbb{H}}(w_1, w) \times \underline{\mathbb{H}}(w_2, w) \xrightarrow{\cong} \underline{\mathbb{H}}(w_1 \oplus w_2, w)$



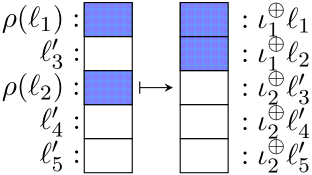
- $\mathbb{H}^{\emptyset} : \mathbb{1} \xrightarrow{\cong} \underline{\mathbb{H}}(\emptyset, w)$

Defragmentation

Complements

For $\rho : w_1 \to w_2$, set $\rho^{\complement} : w_2 \ominus \rho \to w_2$ as the inclusion:

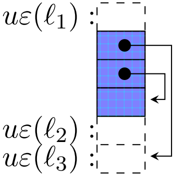
$$w_2 \setminus \rho[w_1] \subseteq w_2$$



$$\underline{\mathbb{H}}(w_2, w) \xrightarrow{\underline{\mathbb{H}}(\cong, w)} \underline{\mathbb{H}}(w_1 \oplus (w_2 \ominus \rho), w)$$

Initialisations

The category $\mathbb{E}$ of initialisations



Objects: Heap layouts/worlds w

Morphisms $\varepsilon : w \to w'$: pairs $\varepsilon = \langle u\varepsilon, \eta_\varepsilon \rangle$ of:

- injection $u\varepsilon : w \to w'$
- **initialisation data** $\eta_\varepsilon \in \underline{\mathbb{H}}(w' \ominus u\varepsilon, w')$

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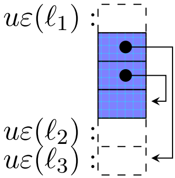

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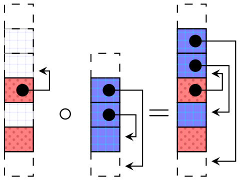
Composition



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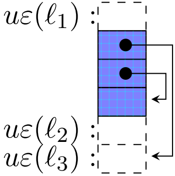

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Heap functor

$$\mathbb{H}w = \underline{\mathbb{H}}(w, w) \cong \mathbb{E}(\emptyset, w)$$

so $\mathbb{H}$ is a representable functor in $\mathbf{E} := [\mathbb{E}, \mathbf{Set}]$

The hiding monad $P : \mathbf{E} \to \mathbf{E}$

Comma category

For the forgetful $u : \mathbb{E} \to \mathbb{W}$, the comma $w \downarrow u$ is:

Objects: $\mathbb{W}$-morphisms $\rho : w \to w'$

Morphisms $\left(\rho_1 \overset{w}{\underset{w_1}{\downarrow}} \right) \xrightarrow{\varepsilon} \left(\rho_2 \overset{w}{\underset{w_2}{\downarrow}} \right)$: Initialisations $w_1' \xrightarrow{\varepsilon} w_2'$ s.t.

$$\begin{tikzcd}
 & w_1 \arrow[d, "u\varepsilon"] \\
w \arrow[ur, "\rho_1"] \arrow[dr, "\rho_2"'] \arrow[r, phantom, "="] & w_2
\end{tikzcd}$$

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w \arrow[ur, "\rho_1"] \arrow[dr, "\rho_2"'] \arrow[r, phantom, "="] & \\
 & w_2
\end{tikzcd}$$

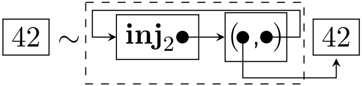
Object map

Define for $A \in \mathbf{E} = [\mathbb{E}, \mathbf{Set}]$:

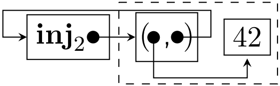
$$PAw := \int^{w \to w' \in w \downarrow u} A$$

$$:= \Big(\sum_{w \to w' \in w \downarrow u} Aw' \Big) / \sim$$

In $[\rho : w \to w', x] \in PAw$ the locations in $w' \ominus \rho$ are private to x



$\in P\mathbb{H}\,\{\ell : \texttt{data}\}$



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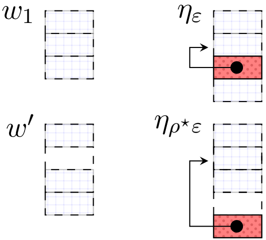
Functorial action of PA

For $\varepsilon : w_1 \to w_2$ in $\mathbb{E}$:

$$\begin{aligned} PA\varepsilon \ : \qquad PAw_1 &\to \qquad\qquad PAw_2 \\ [\rho : w_1 \to w', a] &\mapsto [\varepsilon^\star\rho, A(\rho^\star\varepsilon)(a)] \end{aligned}$$

where

$$\begin{tikzcd} w_1 \arrow[rr, "\varepsilon"] \arrow[d, "\rho"'] \arrow[drr, phantom, "="] & & w_2 \arrow[d, "\varepsilon^\star\rho"] \\ w' \arrow[rr, "\rho^\star\varepsilon"'] & & \rho \oplus_w wu\varepsilon \end{tikzcd}$$



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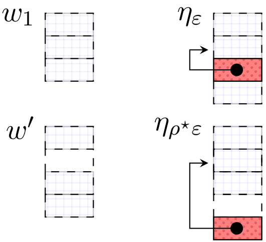

See paper for return, bind, etc.

State transformer

Enrichment

$\mathbf{E} = [\mathbb{E}, \mathbf{Set}]$ is enriched over $\mathbf{W} = [\mathbb{W}, \mathbf{Set}]$ with tensors:

$$X \odot A \coloneqq (X \circ u) \times A$$

$$(A \multimap B)w \coloneqq \int_{w \to w' \in w \downarrow u} Aw' \Rightarrow (Bw')$$

(directly/as a $\mathbf{W}$-actegory [Gordon-Power'97, Janelidze-Kelly'01])

A monad for full ground storage (following [Egger et al.'14])

Take $T \coloneqq \mathbb{H} \multimap P(- \odot \mathbb{H})$

$$\begin{tikzcd}
\mathbf{W} \arrow[rr, bend right, "- \odot \mathbb{H}"'] \arrow[rr, phantom, "\top"] & & \mathbf{E} \arrow[ll, bend right, "\mathbb{H} \multimap -"'] \arrow[loop right, "P"]
\end{tikzcd}$$

Full ground storage monad

For every $X \in \mathbf{W} = [\mathbb{W}, \mathbf{Set}]$:

$$TXw \subseteq \prod_{w \to w' \in \mathbb{W}} \mathbb{H}w' \Rightarrow \left(\sum_{w' \to w'' \in \mathbb{W}} Xw'' \times \mathbb{H}w'' \right) / \sim$$

The monad

$$(TX)w = \int_{w \to w' \in w \downarrow u} \mathbb{H}w' \Rightarrow \left(\int^{w' \to w'' \in w \downarrow u} X \circ uw'' \times \mathbb{H}w'' \right)$$

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Garbage collection

Constant functors

Functors $X \in \mathbf{W}$ whose action $X\rho$ is a bijection.

Theorem (effect masking)

For every pair of constant functors $\Gamma, X \in \mathbf{W}$, *every morphism* $f : \Gamma \to TX$ *factors uniquely through the monadic unit:*

$$\begin{tikzcd} \Gamma \arrow[rr, "f"] \arrow[dr, dashed, "\mathrm{runST}\,f"'] & {} \arrow[d, phantom, "="] & TX \\ & X \arrow[ur, "\mathrm{return}^T"'] & \end{tikzcd}$$

Interprets a multi-monadic-metalanguage with a $\mathbf{runST}$ construct [Launchbury-Peyton Jones'94]

Garbage collection

Proof.

$$
\begin{array}{cl}
\mathbb{1} \longrightarrow \mathbb{H} \multimap P(X \odot \mathbb{H}) & \text{in } \mathbf{W} \\
\hline\hline
\mathbb{H} \longrightarrow P(X \odot \mathbb{H}) & \text{in } \mathbf{E} \\
\hline\hline
\mathbb{H} \longrightarrow X \odot P\mathbb{H} & \text{by lemma} \\
\hline\hline
\mathbb{E}(\emptyset, -) \longrightarrow X \odot P\mathbb{H} & \text{in } \mathbf{E} \\
\hline\hline
\mathbb{1} \longrightarrow (X \odot P\mathbb{H})\emptyset & \text{in } \mathbf{Set}\text{, by Yoneda} \\
\hline\hline
\mathbb{1} \longrightarrow X\emptyset & \text{by lemma} \\
\hline\hline
\mathbb{1} \longrightarrow X & \text{in } \mathbf{W}
\end{array}
$$

and chase a generic morphism upwards. $\square$