A dependently-typed construction of a regular conditional expectation

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slides: 

paper: 

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Goal

Convenient logical foundations for **formal** statistics

Using:

- **dependently-typed** concepts with
- **semantics** that support
- **discrete** and **continuous** probability distributions

while:

- abstracting over bureaucracy ('measurability')

'dependently-typed probabilistic programming'

What have we done?

Semantic **model** for **dependent types** $\Gamma, x : A \vdash B$

best case scenario: (split full comprehension category [Jacobs'93])

$\Gamma \vdash (x : A) \to B$
dependent functions
(Π-types)

$\Gamma \vdash (x : A) \times B$
dependent pairs
(Σ-types)

$\Gamma \vdash M =::= K$
propositional equality
(extensional)

$\Gamma \vdash \{x : A | \varphi\}$
subspaces
(refinement types)

$a, b : \mathbb{R} \vdash (a, b]$
continuous spaces

where **construction** means (quasi-)**measurable map**

What have we done? And why?

Dependent types trim ill-defined edge cases

- **Urn models** indexed by their total number of balls: [Jacobs'22]

$$n : \mathbb{N} \vdash \mathrm{Urn}_n \coloneqq \{(b, r) : \mathbb{N} \times \mathbb{N} | b + r = n\}$$

- insertion:

$$n : \mathbb{N} \vdash \{\mathsf{R}, \mathsf{B}\} \to \mathrm{Urn}_{1+n}$$

What have we done?

Fibred Giry monad (split, in fact) [Lawvere'62, Giry'80]

$\Gamma \vdash \mathbf{D}A$
distributions / measures
(non-normalised)

$\Gamma \vdash \mathbf{P}A \coloneqq \{\mu : \mathbf{D}A | \mu(A) = 1\}$
probability distributions

$(\Gamma \vdash A)$

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supporting continuous probability with the usual properties:
(fibred variant of Kock's ['12] **synthetic measure theory**)

$a, b : \mathbb{R}, a < b \vdash \mathbf{U}_{(a,b]} : \mathbf{P}(a, b]$
uniform distribution

$$\Gamma, \mu : \mathbf{D}A, f : A \to [0,\infty], \nu : \mathbf{D}B, g : B \to [0,\infty] \vdash$$
$$\int \mathrm{d}\mu f \int \mathrm{d}\nu g = \iint (\mu \otimes \nu)(\mathrm{d}x, \mathrm{d}y)(f\,x) \cdot (f\,y) = \int \mathrm{d}\nu g \int \mathrm{d}\mu f$$
(fibred Fubini's theorem)

What have we done? And why?

Dependent types trim ill-defined edge cases

- Drawing from an urn:

$$n : \mathbb{N} \vdash \mathsf{draw} : \mathrm{Urn}_{1+n} \to \mathbf{P}(\mathrm{Urn}_n \times \{\mathsf{R}, \mathsf{B}\})$$

$$\begin{aligned}\mathsf{draw}\,(b, r, p) \coloneqq\ & c \sim \mathbf{B}(\mathsf{lift}\ \tfrac{b}{b+r}\ \mathsf{since}\ b + r = 1 + n > 0) \\ & \mathbf{if} \quad c \qquad\qquad\qquad\qquad \uparrow p \\ & \mathbf{then}\ \boldsymbol{\delta}(\mathsf{lift}\ (b-1, r)\ \mathsf{since}\ b - 1 + r = 1 + n - 1 = n) \\ & \mathbf{else}\ \ \boldsymbol{\delta}(\mathsf{lift}\ (b, r-1)\ \mathsf{since}\ b + r - 1 = 1 + n - 1 = n)\end{aligned}$$

- Sampling with replacement:

$$\begin{aligned}n : \mathbb{N}, u_0 : \mathrm{Urn}_{1+n} \vdash \mathsf{replace} \coloneqq\ & (u_1, b) \sim \mathsf{draw}\, u_0 \\ & \boldsymbol{\delta}(\mathsf{insert}(u_1, b), b) : \mathbf{P}(\mathrm{Urn}_{1+n} \times \{\mathsf{R}, \mathsf{B}\})\end{aligned}$$

replace '**almost-sure** safety' with sure '**safety**'

What have we done?

Dependently-typed probability

Regular conditional expectation inhabiting inherently dependent space:

$$\Gamma \vdash \mathbb{E} : \underset{\text{observation}}{(\mu : \mathbf{D}\Omega)} \to (H : \Omega \to \Theta) \to (f : \underset{\mu\text{-integrable r.v.}}{\mathcal{L}^1(\Omega, \mu)}) \to \underset{\text{conditional expectation}}{\left\{ g : \mathcal{L}^1(\Omega, \mu_H) \middle| g = \mathbb{E}_\mu \left[f \mid H \right] \right\}} \qquad \text{(standard } \Theta\text{)}$$

by extracting **construction** from standard proof

Talk structure

- Motivation
- More context
- semantic foundation
  - Recap: quasi-Borel spaces
  - quasi-Borel families
  - fibred distribution and probability monads
- conditional expectation
  - random variables
  - conditional expectation
  - measurable orthonormality and separability

What have we done?

Best possible situation:

- Semantic model for dependent types (Π, Σ, extensional identity, subspaces) (split fibration)
- Semantic model for dependently-typed probability (split distribution and probability monads)

Case study

- dependently-typed probability (measurability of conditional expectation under regularity assumptions)

Why do it?

Community interest

- Stream-sampling for probabilistic programming using continuous functions via **compactly generated Hausdorf spaces** [Dahlqvist, Silva, and Smith'23]
- **Synthetic probability** for dependent types with Markov fibrations [Rischel'26]
- Probabilistic **graphical models** with the codomain fibration [St Clere Smithe '24, '25]

Personal interest

- Dependently-typed **Probabilistic programming** and **Bayesian inference** (WIP) (use Brady's ['21] Idris 2)
- Unsatisfactory early version of use-case using **HOL** and **refinement types** CIRM tutorial [Kammar'22, unpublished]

Why is it tricky?

Aumann's theorem ['61]

No simply-typed measurable-function measurable space, let alone dependent-functions

Locally Cartesian-closed categories

Codomain fibrations of sheaf models for probability model full dependent types

several models for classical continuous probability

quasi-Borel spaces and their precursor [Staton et al. '16+'17]

probability sheaves [Simpson'17+'24]

nominal techniques [Li, Aytac, Johnson-Freyd, Ahmed, Holtzen'24]

still need a fibred probability monad!

(codomain fibration also not **split**—

inconvenient for external development in Staton et al. and Simpson models)

Why is it tricky?

What should $\Gamma \vdash \mathbf{P}A$ be?

Non-answer:

$$\gamma : \Gamma \vdash \text{✗}A \coloneqq \left\{ \mu : \mathbf{P}(\Gamma, x : A) \middle| \mu(\mathrm{d}\gamma', x \mapsto a)\text{-a.s. } \gamma' = \gamma \right\}$$

✗A is a monad on each fibre, but reindexing/substitution does not preserve it:

$$(\text{✗}A)\,[\theta] \ncong \text{✗}(A\,[\theta])$$

✗A isn't **fibred monad**

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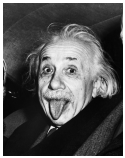
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Danel and I listened to **Einstein**:

> "If you are out to describe the truth, leave elegance to the tailor."
>
> We started developing semantics for dependent types with non-fibred monads.



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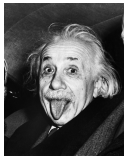
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Danel and I listened to **Einstein**. We should have listened to **Rasmus**:



"If you are out to describe the truth, leave elegance to the tailor."

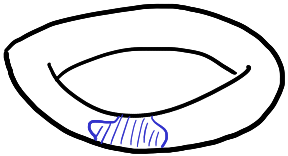
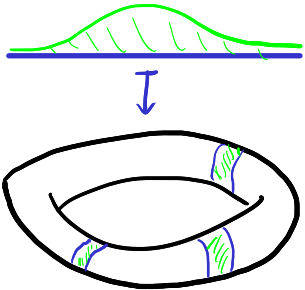
"Danel and Ohad, your pain threshold is much too high for your own good."



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Quasi-Borel spaces recap: core idea [Staton et al.'17]

	measurable spaces	**quasi Borel** spaces
		
primitive notions:	points & **admissible events** (measurable subsets)	points & **admissible random elements**
derived notions:	**admissible random elements** $\alpha : \mathbb{R} \to M$	**admissible events** $E \in \mathcal{B}_X$

Metaphorology[1] $\mathcal{R}$

over set $\underline{X}$ of **points**: subset of functions—**admissible random elements**—$\alpha : \mathbb{R} \to \underline{X}$ closed under:

constants:

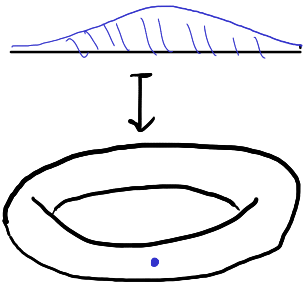
$$\frac{x \in \underline{X}}{\underline{x} \coloneqq (\lambda r.x) \in \mathcal{R}}$$

precomposition:

$$\frac{\alpha \in \mathcal{R} \qquad \varphi \in \mathbf{Meas}(\mathbb{R}, \mathbb{R})}{(\alpha \circ \varphi) \in \mathcal{R}}$$

recombination:

$$\frac{I \subseteq \mathbb{N} \quad E : I \to \mathcal{B}_{\mathbb{R}} \quad \mathbb{R} = \biguplus_{i \in I} E_i \quad \alpha : I \to \mathcal{R}}{[E_i.\alpha_i]_{i \in I} \coloneqq (\lambda r \in E_i.\alpha_i r) \in \mathcal{R}}$$



[1] $\mu\varepsilon\tau\alpha$ ('meta', across) and $\varphi\varepsilon\rho\omega$ ('phero', to carry).

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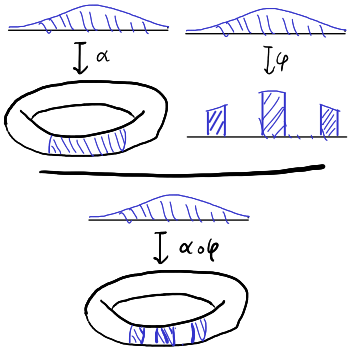
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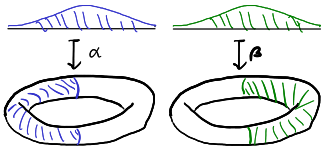
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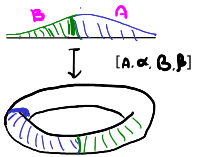
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Quasi-Borel spaces: core definitions [Staton et al.'17]

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Quasi Borel space (qbs) X

set of points $\underline{X}$ equipped with a metaphorology $\mathcal{R}_X$ over it

[1] $\mu\varepsilon\tau\alpha$ ('meta', across) and $\varphi\varepsilon\rho\omega$ ('phero', to carry).

Γ-fibred metaphorology $\mathcal{R}^{-}$ over $\gamma : \underline{\Gamma} \vdash \underline{X}_{\gamma}$

$\mathcal{R}_{\Gamma}$-indexed family of subsets $\boldsymbol{v} : \mathcal{R}_{\Gamma} \vdash \mathcal{R}^{\boldsymbol{v}} \subseteq (r : \mathbb{R}) \Rightarrow \underline{X}_{\boldsymbol{v}\, r}$ closed under:

fibred constants:

$$\frac{\gamma \in \underline{\Gamma} \qquad a \in \underline{X}_{\gamma}}{\underline{a} \coloneqq (\lambda r.a) \in \mathcal{R}^{\underline{\gamma}}}$$

fibred precomposition:

$$\frac{\alpha \in \mathcal{R}^{\boldsymbol{v}} \qquad \varphi \in \mathbf{Qbs}(\mathbb{R}, \mathbb{R})}{(\alpha \circ \varphi) \in \mathcal{R}^{\boldsymbol{v} \circ \varphi}}$$

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Quasi Borel family (qbf) $\Gamma \vdash X$

a family $\gamma : \underline{\Gamma} \vdash \underline{X}_\gamma$ equipped with a Γ-fibred metaphorology $\mathcal{R}_X$ over $\underline{X}$.

Quasi Borel families

Examples

- $\dfrac{\Gamma, X \text{ qbses}}{\Gamma \vdash X \text{ qbf}}$ $\qquad \gamma : \underline{\Gamma} \vdash \underline{X}_\gamma \coloneqq X \qquad \boldsymbol{v} : \mathcal{R}_\Gamma \vdash \mathcal{R}_X^{\boldsymbol{v}} \coloneqq \mathcal{R}_X$
- Intervals are qbfs:

$$a, b : [-\infty, \infty] \vdash [a, b], [a, b), (a, b], (a, b)$$

with $\mathcal{R}^{a \mapsto \alpha, b \mapsto \beta}$ those measurable functions $\varphi \in \mathbf{Qbs}(\mathbb{R}, [0, \infty])$ such that, for all $r \in \mathbb{R}$, $\varphi\, r$ is in the interval delimited by $\alpha\, r$ and $\beta\, r$.

Quasi-measurable maps

Qbf map $(\theta \vdash f) : (\gamma : \Gamma \vdash X_\gamma) \to (\delta : \Delta \vdash Y_\delta)$

a family map $(\theta \vdash f) : (\gamma : \underline{\Gamma} \vdash \underline{X}_\gamma) \to (\delta : \underline{\Delta} \vdash \underline{Y}_\delta)$, i.e.:

$$\theta : \underline{\Gamma} \to \underline{\Delta} \qquad \gamma : \Gamma \vdash f_\gamma : \underline{X}_\gamma \to Y_{\theta\gamma}$$

such that:

- $\theta : \Gamma \to \Delta$ in $\mathbf{Qbs}$, i.e., preserves random elements:

$$\boldsymbol{v} \in \mathcal{R}_\Gamma \implies (\theta \circ \boldsymbol{v}) \in \mathcal{R}_\Delta$$

- f preserves fibred random elements:

$$\mathcal{R}_X^{\boldsymbol{v}} \ni \alpha \qquad \Longrightarrow \qquad (f \overset{\theta}{\circ} \alpha) \coloneqq (\lambda r. f_{\theta r}(\alpha\, r)) \in \mathcal{R}_Y^{\theta \circ \boldsymbol{v}}$$

Type-theory model

Theorem

The category of qbfs and their maps, $\mathbf{Qbf}$*, equipped with the functor:*

$$\mathsf{Index} : (\Gamma \vdash X_) \mapsto \Gamma : \mathbf{Qbf} \to \mathbf{Qbs}$$

is a split fibred category.
It supports:

- *split* Π*-types;*
- Σ*-types;*
- *split extensional propositional equality; and*
- *split subspaces.*
- *split distribution and probability monads:*
  - *conservative extension of classical probability and non-fibred case*
  - *fibred model of Kock's ['12] synthetic measure theory*

The **Grothendieck construction** *and the* **preimage** *form a fibred adjoint equivalence between* Index *and the codomain fibration* $\mathbf{Qbs}^{\to} \to \mathbf{Qbs}$*.*

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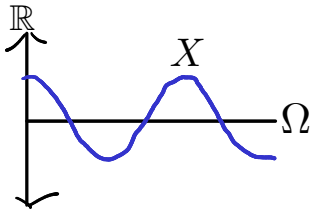
Random variables

Fibred probability space $\Gamma \vdash \langle \Omega, \mu \rangle$

- fibred space $\Gamma \vdash \Omega$
- fibred probability distribution $\Gamma \vdash \mu : \mathbf{P}\Omega$

Fibred random variable $\Gamma \vdash X : \mathbf{RV}(\Omega, \mathbf{D})$ in a probability space $\Gamma \vdash \mu : \Omega$

any function $\Gamma \vdash X : \Omega \to [-\infty, \infty]$



$\Omega := [0, 1] \qquad X_\omega := \cos(3\pi\omega)$

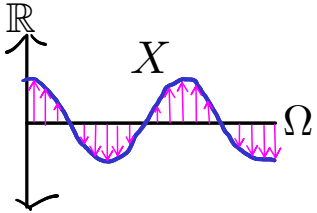
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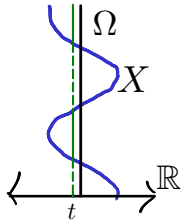
Law of a random variable

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the push-forward measure:

$$\Pr_{x \sim \mu_X}[x \in E] = \Pr_{\omega \sim \mu}[X_\omega \in E]$$

Example ($X_\omega := \cos(3\pi\omega)$)



Random variables

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- fibred probability distribution $\Gamma \vdash \mu : \mathbf{P}\Omega$

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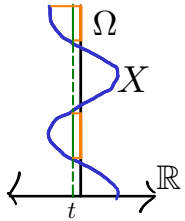
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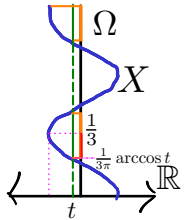
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$$\begin{aligned}&\mathrm{CDF}(\mu_X, t)\\ &= 3\left(\tfrac{1}{3} - \tfrac{1}{3\pi}\arccos t\right)\\ &= 1 - \tfrac{1}{\pi}\arccos t\end{aligned}$$

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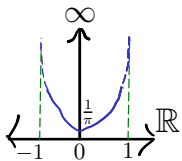
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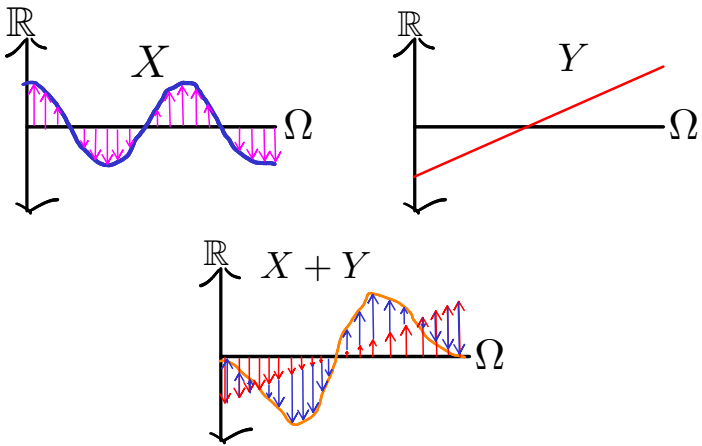


$$\begin{aligned}\mathrm{PDF}(\mu_X; x) &:= \frac{\mathrm{d}\mathbf{U}[0,1]_X}{\mathrm{d}\lambda}(x) \\ &= \frac{\mathrm{d}(1 - \frac{1}{\pi}\arccos t)}{\mathrm{d}t}(x) \\ &= \frac{1}{\pi\sqrt{1-x^2}}\end{aligned}$$

Random variables

Operations on random variables

- $\Gamma \vdash \mathbf{RV}(\Omega, \mu)$ is a (fibred) real vector space.



Random variables

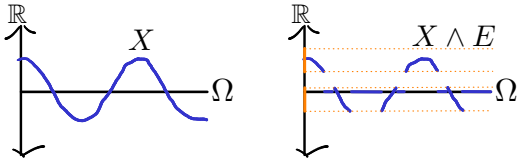
Operations on random variables

- $\Gamma \vdash \mathbf{RV}(\Omega, \mu)$ is a (fibred) real vector space.
- **clamping** to a Borel subset:

$$\Gamma, X : \mathbf{RV}(\Omega, \mu), E : \mathcal{B}_{[-\infty,\infty]} \vdash X \wedge E \coloneqq \lambda\omega. \begin{cases} \omega \in E : & X_\omega \\ \omega \notin E : & 0 \end{cases} : X \to \{0\} \cup E$$

NB: Borel subsets space

$$\frac{\Gamma \vdash X}{\Gamma \vdash \mathcal{B}_X \coloneqq (X \to \mathbf{Bool})}$$



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- **negative fragment**, similarly:

$$\Gamma, X : \mathbf{RV}(\Omega, \mu) \vdash X^- \coloneqq -(X \wedge [-\infty, 0]) : \Omega \to [0, \infty]$$

so that: $\Gamma, X : \mathbf{RV}(\Omega, \mu) \vdash X =::= X^+ - X^-$

Lebesgue integrability

Integrating RVs

We can integrate RVs with at least one finite fragment:

$$\Gamma \vdash \int \mathrm{d} : (\mu : \mathbf{P}\Omega) \times \left\{ X : \mathbf{RV}(\mathbf{P}\Omega, \mu) \middle| \infty > \int \mathrm{d}\mu X^+ \vee \infty > \int \mathrm{d}\mu X^- \right\} \to [0, \infty]$$

$$\int \mathrm{d}\mu X \coloneqq (\int \mathrm{d}\mu X^+) - (\int \mathrm{d}\mu X^-)$$

Lebesgue integrable random variables $\Gamma, p : [1, \infty) \vdash \mathcal{L}^p(\Omega, \mu)$

$$\mathcal{L}^p(\Omega, \mu) \coloneqq \left\{ X \in \mathbf{RV}(\Omega, \mu) \middle| \infty > \int \mathrm{d}\,|X|^p \right\}$$

NB: $\mathrm{L}^p(\Omega, \mu) \coloneqq \mathcal{L}^p(\Omega, \mu)/(\mu\text{-a.s. equality})$

we work with $\mathcal{L}^p$, not L^p!

Functional analysis of the Lebesgue spaces

Each Lebesgue space $\mathcal{L}^p(\Omega, \mu)$:

- sub-vector space $\mathcal{L}^p(\Omega, \mu) \subseteq \mathbf{RV}(\Omega, \mu)$

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'semi': $\|X\|_p = 0$ iff μ-a.s. $X = 0$.

The semi-norm is complete: every Cauchy-sequence has a limit, measurably:

$$\Gamma, p : [1, \infty) \vdash$$

$$\mathsf{Cauchy}_p(\Omega, \mu) \coloneqq \left\{ X : \mathbb{N} \to \mathcal{L}^p(\Omega, \mu) \middle| \forall \varepsilon. \exists n. \forall k, \ell > n. \, \|X_k - X_m\|_p < \varepsilon \right\}$$

$$\lim^p : (X : \mathsf{Cauchy}_p(\Omega, \mu)) \to \left\{ L : \mathcal{L}^1(\Omega, \mu) \middle| \forall \varepsilon. \exists n. \forall k > n. \, \|X_k - L\|_p < \varepsilon \right\}$$

$$\lim^p X \coloneqq \lambda\omega. \begin{cases} \mathsf{Cauchy}(\lambda n. X_{n,\omega}) : & \lim_n X_{n,\omega} \\ \text{otherwise} : & 0 \end{cases}$$

and overall we have a Banach-space-up-to-a.s.-equality

Functional analysis of the Lebesgue spaces

For $\mathcal{L}^2$, the space has a geometry:

- inner product:

$$\Gamma, p : [1, \infty) \vdash, X, Y : \mathcal{L}^2(\Omega, \mu) \vdash \langle X, Y \rangle \coloneqq \int \mu(\mathrm{d}\omega) X_\omega \cdot Y_\omega$$

and overall we have a Hilbert-space-up-to-a.s.-equality.

Conditional expectation

Central higher-order concept to modern probability:

$$\frac{\Gamma \vdash \mu : \mathbf{D}\Omega \qquad \Gamma \vdash H : \Omega \to \Theta \qquad \Gamma \vdash f : \mathcal{L}^1(\Omega, \mu) \qquad \Gamma \vdash g : \mathcal{L}^1(\Theta, \mu_H)}{\Gamma \vdash (g = \mathbb{E}_\mu[f|H]) :\equiv \quad \forall w : \Theta \to \mathbb{W}.}$$

$$\int \mu_H(\mathrm{d}\theta) g(\theta) \cdot w(\theta) = \int \mu(\mathrm{d}\omega) f(\omega) \cdot w(H\,\omega) : \mathbf{Prop}$$

observation — latent rv — observed rv

- The conditional expectation 'integrates' unobserved outcomes
- Statistics (e.g., mean, standard deviation and other moments) of $g = \mathbb{E}_\mu[f \mid H]$ agree with those of $f \circ H$ in H.

Conditional expectation

Theorem

Let $\Gamma \vdash \Omega$ be any qbf and $\vdash \Theta$ a standard Borel space. We have a measurable dependent function:

$$\Gamma \vdash \mathbb{E} : (\mu : \mathbf{P}\Omega) \to (H : \Omega \to \Theta) \to (f : \mathcal{L}^1(\Omega, \mu)) \to \left\{ g : \mathcal{L}^1(\Theta, \mu_H) \middle| g = \mathbb{E}_\mu\left[f|H\right] \right\}$$

Proof consists of noticing the 'constructive' content in the standard proof [Williams'91]

What have we done?

Best possible situation:

- Semantic model for dependent types (Π, Σ, extensional identity, subspaces)
  (split fibration)
- Semantic model for dependently-typed probability
  (split distribution and probability monads)

Case study

- dependently-typed probability
  (measurability of conditional expectation under regularity assumptions)

Future work

Dependently-typed accounts of:

- disintegration;
- martingale theory;
- stochastic differential equations
  (NB: generalized conditional expectation may need further assumptions)
- universe constructions
  (already have $\mathbf{Sbs} \coloneqq \mathcal{B}_{\mathbb{R}}$; $\mathbf{Pol}$ and $\mathbf{CSepMet}$ via Effros space)
- implications to Bayesian inference implementations;
- implications to probabilistic programming semantics
  (operational and denotational)

Thank you!