

A monad for full ground reference cells

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joint work with

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Dynamic allocation of memory cells

Kinds of storage

type data = **bool**

type linked_list = **1** + **ref** list_cell

type list_cell = **ref** data * **ref** linked_list

type process = **1** + (**1** → **ref** process)

ground

full ground:
refs to refs
cyclic data

general:
suspended computation
effectful functions

This talk:

full ground storage only

Local state semantics

	full			
	ground	ground	general	
operational	✓	✓	✓	syntactic, ubiquitous dangling pointers, e.g.: Benton, Bohr, Hofmann, Birkedal et al. '06
relational	✓	✓	✓	
step indexing	✓	✓	✓	syntactic or relational Ahmed '04 onwards Birkedal et al. '10 onwards
game	✓	✓	✓	CBN-ish, partial dangling pointers e.g. Reddy-Yang '04
parametricity	✓	✓		
possible-world	→ ✓	→ ✓	→	▶ Sets-with-structure and structure preserving functions ▶ Monad over a bi-CCC ▶ Avoid dangling pointers
This work Levy'02 Reynolds and Oles'82, Moggi'90, O'Hearn and Tennent'92, Stark'94, Ghica'97, Plotkin-Power'02				

Goal

Semantics for full ground storage:

- ▶ Sets-with-structure and structure preserving functions
- ▶ Possible-world semantics
- ▶ Monad over a bi-CCC
- ▶ Avoid dangling pointers

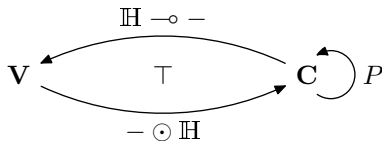
Evaluation

- ▶ Effect masking
- ▶ Adequacy
- ▶ Local ground storage equations

A state transformer

A monad for full ground storage

Take $T \stackrel{\text{def}}{=} \mathbb{H} \multimap P(- \odot \mathbb{H})$



Talk structure

- ▶ Full ground storage
- ▶ Worlds and initialisations
- ▶ A monad for hiding/encapsulation
- ▶ The monad, explicitly
- ▶ Evaluation
- ▶ Conclusion

Full ground storage

Full ground signature $\langle \mathbb{m}, \mathbf{type} \rangle$

- ▶ Countably many $c \in \mathbb{m}$ **storable-type names**
- ▶ A **full ground type interpretation** $\mathbf{type} : \mathbb{m} \rightarrow \mathbb{G}$

Where the **full ground types** $\gamma \in \mathbb{G}$ are:

$$\gamma ::= \mathbf{0} \mid \gamma_1 + \gamma_2 \mid \mathbf{1} \mid \gamma_1 * \gamma_2 \mid \mathbf{ref} \ c$$

Example

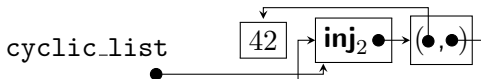
$\mathbb{m} \stackrel{\text{def}}{=} \{\text{data}, \text{linked_list}, \text{list_cell}\}$

```
type data      = bool  
type linked_list = 1 + ref list_cell  
type list_cell  = ref data * ref linked_list
```

Simultaneous allocation

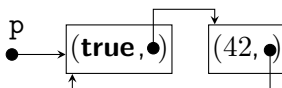
► Examples:

```
letref (payload      : ref data      ) := 42,  
      (cyclic_list: ref linked_list) := inj2head,  
      (head       : ref list_cell  ) := (payload, cyclic_list)  
in cyclic_list
```



```
► letref (p: ref bit_cell) := (true, q),  
      (q: ref int_cell) := (42, p),  
in p
```

type bit_cell = **bool** * **ref** int_cell
type int_cell = **int** * **ref** bit_cell



non-sequentialisable

A call-by-value calculus: λ_{ref} (syntax)

$\tau ::=$	types
ref c	reference
0	empty
$\tau_1 + \tau_2$	binary sum
1	unit
$\tau_1 * \tau_2$	product
$\tau_1 \rightarrow \tau_2$	function
$v ::=$	values
ℓ	location ($\ell \in \mathbb{L}$)
x	identifier
inj $_i^{\tau_1 + \tau_2} v$	constructor
$()$	unit
(v_1, v_2)	pair
$\lambda x : \tau. t$	function

Sugar

let $(x : \tau) = t$ **in** $s \equiv (\lambda x : \tau. s) t$

new $_c t \equiv$

let $(x : \text{type } c) = t$

in letref $(y : \text{ref } c) := x$ **in** y

$t, s ::=$	terms
ℓ	location ($\ell \in \mathbb{L}$)
x	identifier
inj $_i^{\tau_1 + \tau_2} t$	sum constructor
$()$	unit
(t, s)	pair
$\lambda x : \tau. t$	function abstraction
match t with $\{\}^\tau$	pattern matching
match t with $\{\text{inj}_1 x_1 \mapsto s_1$ $\text{inj}_2 x_2 \mapsto s_2\}$	
prj $_i t$	projections
$t s$	function application
$t := s$	assignment
! t	dereferencing
letref $(x_1 : \text{ref } c_1) := v_1,$ $\vdots$ $(x_n : \text{ref } c_n) := v_n$	allocation
in t	

A call-by-value calculus: λ_{ref} (type system)

Typing judgements $\boxed{\Gamma \vdash_w t : \tau}$

Type contexts: $\Gamma = x_1 : \tau_1, \dots, x_n : \tau_n$

Heap layouts: $w = \{\ell_1 : c_1, \dots, \ell_n : c_n\}$

$\underline{w} \stackrel{\text{def}}{=} \{\ell_1, \dots, \ell_n\}$

$$\frac{(\ell : c) \in w}{\Gamma \vdash_w \ell : \mathbf{ref} \ c}$$

$$\frac{\Gamma \vdash_w t : \mathbf{ref} \ c \quad \Gamma \vdash_w s : \mathbf{type} \ c}{\Gamma \vdash_w t := s : \mathbf{1}}$$

$$\frac{\Gamma \vdash_w t : \mathbf{ref} \ c}{\Gamma \vdash_w !t : \mathbf{type} \ c}$$

for $i = 1, \dots, n$: $\Gamma, x_1 : \mathbf{ref} \ c_1, \dots, x_n : \mathbf{ref} \ c_n \vdash_w v_i : \mathbf{type} \ c_i$

$$\frac{\Gamma, x_1 : \mathbf{ref} \ c_1, \dots, x_n : \mathbf{ref} \ c_n \vdash_w t : \tau}{\Gamma \vdash_w \mathbf{letref}(x_1 : \mathbf{ref} \ c_1) := v_1,$$

$\vdots$

$(x_n : \mathbf{ref} \ c_n) := v_n \ \mathbf{in} \ t : \tau$

A call-by-value calculus: λ_{ref} (operational semantics)

$$\eta = \langle v_\ell \rangle_{\ell \in \underline{w}_\eta}$$

$$\overline{\langle \ell, \eta \rangle \Downarrow \langle \ell, \eta \rangle}$$

$$\frac{\langle t, \eta \rangle \Downarrow \langle \lambda x : \tau. t', \eta_1 \rangle \quad \langle s, \eta_1 \rangle \Downarrow \langle v', \eta_2 \rangle \quad \langle t'[x \mapsto v'], \eta_2 \rangle \Downarrow \langle v, \eta' \rangle}{\langle t \ s, \eta \rangle \Downarrow \langle v, \eta' \rangle}$$

$$\frac{\langle t, \eta \rangle \Downarrow \langle v, \eta' \rangle}{\langle \text{inj}_i^{\tau_1 + \tau_2} t, \eta \rangle \Downarrow \langle \text{inj}_i^{\tau_1 + \tau_2} v, \eta' \rangle}$$

$$\overline{\langle (), \eta \rangle \Downarrow \langle (), \eta \rangle}$$

$$\frac{\langle t, \eta \rangle \Downarrow \langle \ell, \hat{\eta} \rangle \quad \langle s, \hat{\eta} \rangle \Downarrow \langle v, \eta' \rangle}{\langle t := s, \eta \rangle \Downarrow \langle (), \eta'[\ell \mapsto v] \rangle} (\ell \in \underline{w}_{\eta'})$$

$$\frac{\langle t_1, \eta \rangle \Downarrow \langle v_1, \hat{\eta} \rangle \quad \langle t_2, \hat{\eta} \rangle \Downarrow \langle v_2, \eta' \rangle}{\langle (t_1, t_2), \eta \rangle \Downarrow \langle (v_1, v_2), \eta' \rangle}$$

$$\frac{\langle t, \eta \rangle \Downarrow \langle \ell, \eta' \rangle}{\langle !t, \eta \rangle \Downarrow \langle \eta'(\ell), \eta' \rangle} (\ell \in \underline{w}_{\eta'})$$

$$\overline{\langle \lambda x : \tau. t, \eta \rangle \Downarrow \langle \lambda x : \tau. t, \eta \rangle}$$

$$\langle t, \eta \rangle \Downarrow \langle \text{inj}_i^{\tau_1 + \tau_2} \hat{v}, \hat{\eta} \rangle$$

$$\langle s_i[x_i \mapsto \hat{v}], \hat{\eta} \rangle \Downarrow \langle v, \eta' \rangle$$

$$\text{match } t \text{ with}$$

$$\left\langle \begin{array}{l} \{ \text{inj}_1 x_1 \mapsto s_1, \eta \} \\ | \text{inj}_2 x_2 \mapsto s_2 \} \end{array} \right\rangle \Downarrow \langle v, \eta' \rangle$$

$$\frac{\langle t, \eta \rangle \Downarrow \langle (v_1, v_2), \eta' \rangle}{\langle \text{prj}_i t, \eta \rangle \Downarrow \langle v_i, \eta' \rangle}$$

$$\frac{\langle t[\theta], \eta \left[\ell_i \mapsto v_i[\theta] \right]_{i=1}^n \rangle \Downarrow \langle v, \eta' \rangle \quad \text{where } \theta \stackrel{\text{def}}{=} [x_i \mapsto \ell_i]_{i=1}^n}{\text{letref}} (\# \underline{w}_\eta \langle \ell_1, \dots, \ell_n \rangle)$$

$$\left\langle \begin{array}{l} (x_1 : \text{ref } c_1) := v_1, \\ \vdots \\ (x_n : \text{ref } c_n) := v_n \text{ in } t \end{array} \right\rangle \Downarrow \langle v, \eta' \rangle$$

A call-by-value calculus: λ_{ref} (meta-theory)

Functoriality of types

If $w \leq w'$ as partial functions, then $\Gamma \vdash_w t : \tau \implies \Gamma \vdash_{w'} t : \tau$. So

$$\begin{array}{lll} \tau^v w \subseteq \tau^v w' & \text{and} & \tau^t w \subseteq \tau^t w' \quad \text{where} \\ \tau^v w \stackrel{\text{def}}{=} \{v \mid \vdash_w v : \tau\} & \text{and} & \tau^t w \stackrel{\text{def}}{=} \{t \mid \vdash_w t : \tau\} \end{array}$$

Non-functoriality of heaps

For $\mathbf{H}w := \{\eta \mid \underline{w}_\eta = \underline{w}, \forall (\ell : c) \in w. \vdash_w \eta(\ell) : \mathbf{type} \ c\}$ and $w \leq w'$ we **lack** functions

$$\mathbf{H}w \rightarrow \mathbf{H}w' \quad \text{and} \quad \mathbf{H}w \leftarrow \mathbf{H}w'$$

A call-by-value calculus: λ_{ref} (meta-theory)

Non-functoriality of heaps

For $\mathbf{H}w := \{\eta \mid \underline{w}_\eta = \underline{w}, \forall (\ell : c) \in w. \vdash_w \eta(\ell) : \mathbf{type} \ c\}$ and $w \leq w'$ we **lack** functions

$$\mathbf{H}w \rightarrow \mathbf{H}w' \quad \text{and} \quad \mathbf{H}w \leftarrow \mathbf{H}w'$$

as we lack **initialisation data** and want to avoid **dangling references**:

$$\begin{aligned} \langle \rangle \in \mathbf{H}\emptyset &\mapsto ? \in \mathbf{H}\{\ell : \text{ptr}\} \\ ? \in \mathbf{H}\{\ell_1 : \text{ptr}\} &\leftarrow \left\langle \begin{array}{l} \ell_1 \mapsto \ell_2, \\ \ell_2 \mapsto \ell_1 \end{array} \right\rangle \in \mathbf{H}\{\ell_1, \ell_2 : \text{ptr}\} \end{aligned}$$

for $\mathbf{type} \ \text{ptr} = \mathbf{ref} \ \text{ptr}$.

A call-by-value calculus: λ_{ref} (meta-theory)

Theorem (totality)

All well-typed closed programs fully evaluate:

$$\begin{aligned} \forall \vdash_w t : \tau, w_1 \geq w, \eta_1 \in \mathbf{H}w_1. \\ \implies \exists w_2 \geq w_1, \vdash_{w_2} v : \tau, \eta_2 \in \mathbf{H}w_2. \langle t, \eta_1 \rangle \Downarrow \langle v, \eta_2 \rangle \end{aligned}$$

Proof highlights

Use Tait's method with Kripke logical predicates, and a w -indexed predicate on $\mathbf{H}w$ that is **not** Kripke.

Allocation case: initialisation data $\vdash_{w \oplus \{\ell_1 : c_1, \dots, \ell_n : c_n\}} v_1, \dots, v_n$ needs to extend $\eta \in \mathbf{H}w_1, w_1 \geq w$:

$$\begin{array}{ccc} w & \leq & w \oplus \{\ell_1 : c_1, \dots, \ell_n : c_n\} \\ \mid \wedge & & \mid \wedge \\ w_1 & \leq & w_1 \oplus \{\ell_1 : c_1, \dots, \ell_n : c_n\} \end{array}$$

Talk structure

- ▶ Full ground storage
- ▶ **Worlds and initialisations**
- ▶ A monad for hiding/encapsulation
- ▶ The monad, explicitly
- ▶ Evaluation
- ▶ Conclusion

The category $\mathbb{W}$

Objects: layouts w

Morphisms $\rho : w \rightarrow w'$: label preserving injections $\rho : \underline{w} \rightarrow \underline{w'}$

Interpreting full ground types

Set $\mathbf{V} := [\mathbb{W}, \mathbf{Set}]$ and define:

$$\llbracket \mathbf{0} \rrbracket \stackrel{\text{def}}{=} 0 \qquad \llbracket \gamma_1 + \gamma_2 \rrbracket \stackrel{\text{def}}{=} \llbracket \gamma_1 \rrbracket + \llbracket \gamma_2 \rrbracket$$

$$\llbracket \mathbf{1} \rrbracket \stackrel{\text{def}}{=} 1 \qquad \llbracket \gamma_1 \times \gamma_2 \rrbracket \stackrel{\text{def}}{=} \llbracket \gamma_1 \rrbracket \times \llbracket \gamma_2 \rrbracket$$

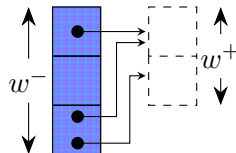
$$\llbracket \mathbf{ref} \ c \rrbracket w \stackrel{\text{def}}{=} \{ \ell \in w \mid w(\ell) = c \} \qquad \llbracket \mathbf{ref} \ c \rrbracket \rho(\ell) \stackrel{\text{def}}{=} \rho(\ell)$$

Semantic heaps

Heaplets

Define $\underline{\mathbb{H}} : \mathbb{W}^{\text{op}} \times \mathbb{W} \rightarrow \mathbf{Set}$:

$$\underline{\mathbb{H}}(w^-, w^+) \stackrel{\text{def}}{=} \prod_{(\ell:c) \in w^-} \llbracket \mathbf{type} \ c \rrbracket w^+$$



Heaps

$$\mathbb{H}w \stackrel{\text{def}}{=} \underline{\mathbb{H}}(w, w)$$

mixed
variance

Combining store fragments

Independent coproducts [Simpson'17]

Given $\# : \mathbb{N} \xrightarrow{\cong} \mathbb{L}$, define:

$$\#w \stackrel{\text{def}}{=} \min\{n \in \mathbb{N} \mid \forall i \geq n. \#i \notin w\}$$

$$\underline{w_1} \oplus \underline{w_2} \stackrel{\text{def}}{=} \underline{w_1} \cup \{\#(\#w_1 + n) \mid \#n \in w_2\}$$

$$\iota_i^\oplus : \underline{w_i} \rightarrow \underline{w_1 \oplus w_2}$$

$$\iota_1^\oplus : \ell \mapsto \ell$$

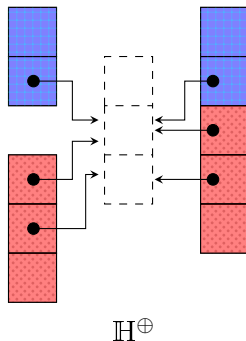
$$\iota_2^\oplus : \#n \mapsto \#(\#w_1 + n)$$

$$(w_1 \oplus w_2)(\iota_i^\oplus \ell) \stackrel{\text{def}}{=} w_i(\ell)$$

and we have: $w_1 \xrightarrow{\iota_1^\oplus} w_1 \oplus w_2 \xleftarrow{\iota_2^\oplus} w_2$ in $\mathbb{W}$

Canonical isomorphisms

$$\mathbb{H}^\oplus : \underline{\mathbb{H}}(w_1, w) \times \underline{\mathbb{H}}(w_2, w) \xrightarrow{\cong} \underline{\mathbb{H}}(w_1 \oplus w_2, w) \quad \mathbb{H}^\emptyset : \mathbb{1} \xrightarrow{\cong} \underline{\mathbb{H}}(\emptyset, w)$$



Initialisations

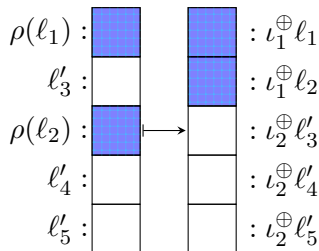
Complements

Given $\rho : w_1 \rightarrow w_2$, define:

$$\underline{w_2 \ominus \rho} \stackrel{\text{def}}{=} \underline{w_2} \setminus \text{Im}(\rho)$$

$$w_2 \ominus \rho(\ell) \stackrel{\text{def}}{=} w_2(\ell)$$

$$\begin{aligned} \rho^{\complement} : w_2 \ominus \rho &\rightarrow w_2 \\ \ell &\mapsto \ell \end{aligned}$$

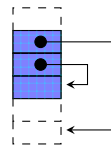


$$\underline{\mathbb{H}}(w_2, w) \xrightarrow{\underline{\mathbb{H}}(\cong, w)} \underline{\mathbb{H}}(w_1 \oplus (w_2 \ominus \rho), w)$$

The category $\mathbb{E}$ of initialisations

Objects: layouts w

Morphisms $\varepsilon : w \rightarrow w'$: injections $u\varepsilon : w \rightarrow w'$ and
initialisation data $\eta_\varepsilon \in \underline{\mathbb{H}}(w' \ominus u\varepsilon, w')$

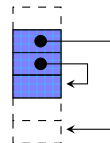


Initialisations

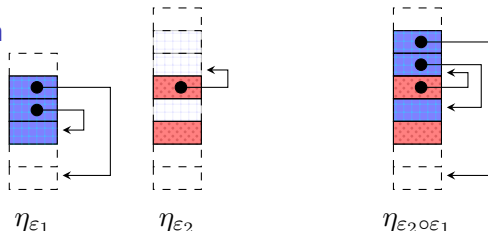
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Composition



Functoriality of heaps

representable functor

$$\mathbb{H}w = \underline{\mathbb{H}}(w, w) \cong \mathbb{E}(\emptyset, w)$$

in $\mathbf{C} \stackrel{\text{def}}{=} [\mathbb{E}, \mathbf{Set}]$

Talk structure

- ▶ Full ground storage
- ▶ Worlds and initialisations
- ▶ **A monad for hiding/encapsulation**
- ▶ The monad, explicitly
- ▶ Evaluation
- ▶ Conclusion

The hiding monad

Comma category

For the forgetful $u : \mathbb{E} \rightarrow \mathbb{W}$, the comma $w \downarrow u$ is:

Objects: $\mathbb{W}$ -morphisms $\rho : w \rightarrow w'$

Morphisms $(\rho_1 \downarrow_{w_1}^w) \xrightarrow{\varepsilon} (\rho_2 \downarrow_{w_2}^w)$: Initialisations $w'_1 \xrightarrow{\varepsilon} w'_2$ s.t. $w \begin{matrix} \xrightarrow{\rho_1} w'_1 \\ \xrightarrow{\rho_2} w'_2 \end{matrix} \begin{matrix} \downarrow u\varepsilon \\ \end{matrix} w_2$

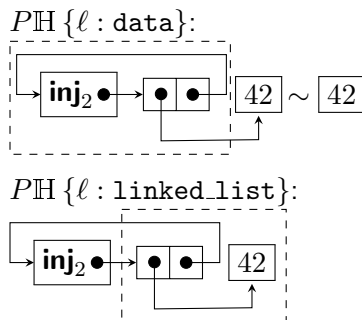
Object map

Define for $A \in \mathbf{C} = [\mathbb{E}, \mathbf{Set}]$:

$$PAw \stackrel{\text{def}}{=} \int^{w \rightarrow w' \in w \downarrow u} A$$

$$\stackrel{\text{def}}{=} \left(\sum_{w \rightarrow w' \in w \downarrow u} Aw' \right) / \sim$$

In $[\rho : w \rightarrow w', x] \in PAw$ the locations in $w' \ominus \rho$ are private to x



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$$\begin{array}{ccc} w & \xrightarrow{\rho_1} & w_1 \\ & \searrow \scriptstyle = & \downarrow \scriptstyle u\varepsilon \\ & \xrightarrow{\rho_2} & w_2 \end{array}$$

Object map

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In $[\rho : w \rightarrow w', x] \in PAw$ the locations in $w' \ominus \rho$ are private to x

Subtleties [standard]

- ▶ $PAw = \text{Colim}_{w \rightarrow w' \in w \downarrow u} Aw'$
convenient coend notation
- ▶ The injections $\rho : w \rightarrow w'$
define the quotient

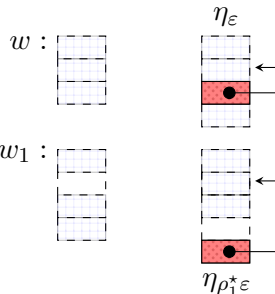
The initialisation $\rho^\star \varepsilon$

Local independent coproducts

Given $w_1 \xleftarrow{\rho_1} w \xrightarrow{\rho_2} w_2$, define:

$$\rho_1 \oplus_w \rho_2 := w \oplus (w_1 \ominus \rho_1) \oplus (w_2 \ominus \rho_2)$$

$$\begin{array}{ccc}
 w & \xrightarrow{\rho_2} & w_2 \\
 \rho_1 \downarrow & = & \downarrow \rho_1^\star \rho_2 \\
 w_1 & \xrightarrow{\rho_1^\star \rho_2} & \rho_1 \oplus_w \rho_2
 \end{array}$$



when $\rho_2 = u\varepsilon$ there is an initialisation $\rho_1^\star \varepsilon : w_1 \rightarrow \rho_1 \oplus_w \rho_2$ with $u\rho_1^\star \varepsilon = \rho_1^\star u\varepsilon$.

Semantic counterpart of

$$\begin{array}{c}
 w \leq w \oplus \{\ell_1 : c_1, \dots, \ell_n : c_n\} \\
 | \wedge \qquad \qquad \qquad | \wedge \\
 w' \leq w' \oplus \{\ell_1 : c_1, \dots, \ell_n : c_n\}
 \end{array}$$

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Functorial action

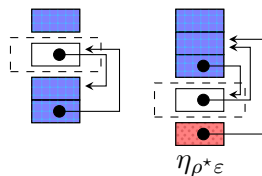
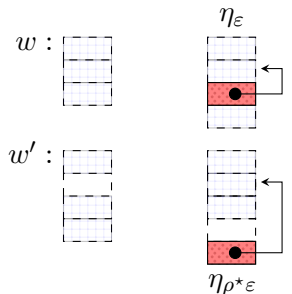
For $\varepsilon : w_1 \rightarrow w_2$ in $\mathbb{E}$:

$$PA\varepsilon : PAw_1 \rightarrow PAw_2$$

$$[\rho : w_1 \rightarrow w', a] \mapsto [\varepsilon^* \rho, A(\rho^* \varepsilon)(a)]$$

where

$$\begin{array}{ccc} w_1 & \xrightarrow{\varepsilon} & w_2 \\ \rho \downarrow & = & \downarrow \varepsilon^* \rho \\ w' & \xrightarrow{\rho^* \varepsilon} & \rho \oplus_w u\varepsilon \end{array}$$



$$P\mathbb{H}w_1 \xrightarrow{P\mathbb{H}\varepsilon} P\mathbb{H}w_2$$

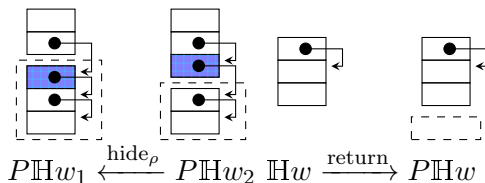
The hiding monad

Object map

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$$\stackrel{\text{def}}{=} \left(\sum_{w \rightarrow w' \in w \downarrow u} Aw' \right) / \sim$$



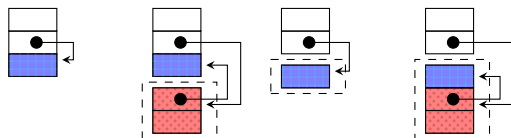
Hiding

For $\rho : w_1 \rightarrow w_2$ we have

$$\text{hide}_\rho : PAw_1 \leftarrow PAw_2$$

Return

$$\text{return}_w a \stackrel{\text{def}}{=} [\text{id}_w, a]$$



Bind

For $g : A \rightarrow PB$, define:

$$([\rho : w \rightarrow w', a] \gg g) \stackrel{\text{def}}{=} \text{hide}_\rho(g_{w'}(a))$$

A state transformer

Enrichment

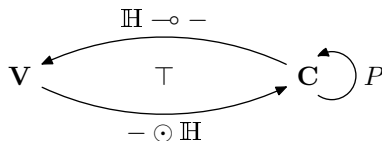
$\mathbf{C} = [\mathbb{E}, \mathbf{Set}]$ is enriched over $\mathbf{V} = [\mathbb{W}, \mathbf{Set}]$ with tensors:

$$X \odot A \stackrel{\text{def}}{=} (X \circ u) \times A$$
$$(A \multimap B)w \stackrel{\text{def}}{=} \int_{w \rightarrow w' \in w \downarrow u} Aw' \Rightarrow (Bw')$$

(directly/as a $\mathbf{V}$ -actegory [Gordon-Power'97, Janelidze-Kelly'01])

A monad for full ground storage

Take $T \stackrel{\text{def}}{=} \mathbb{H} \multimap P(- \odot \mathbb{H})$



Talk structure

- ▶ Full ground storage
- ▶ Worlds and initialisations
- ▶ A monad for hiding/encapsulation
- ▶ **The monad, explicitly**
- ▶ Evaluation
- ▶ Conclusion

The full ground storage monad

Outline

For every $X \in \mathbf{V} = [\mathbb{W}, \mathbf{Set}]$:

$$TXw \subseteq \prod_{w \rightarrow w' \in \mathbb{W}} \mathbb{H}w' \Rightarrow \left(\sum_{w' \rightarrow w'' \in \mathbb{W}} Xw'' \times \mathbb{H}w'' \right) / \sim$$

The monad

$$(TX)w = \int_{w \rightarrow w' \in w \downarrow u} \mathbb{H}w' \Rightarrow \left(\int_{w' \rightarrow w'' \in w' \downarrow u} X \circ uw'' \times \mathbb{H}w'' \right)$$

Subtleties

- ▶ $\int_{w' \rightarrow w'' \in w' \downarrow u} X \circ uw'' \times \mathbb{H}w''$ is contravariant in w' , but the end is taken w.r.t. the covariant action of P
- ▶ The functorial action $TX\rho$ is given via the end

Analogous to $T_{\text{ground}}X\rho$ in the ground case [cf. Plotkin-Power'02]:

$$T_{\text{ground}}Xw = \mathbb{H}_{\text{ground}}w \Rightarrow \int^{w \rightarrow w' \in w / \mathbb{W}} X \times \mathbb{H}_{\text{ground}}$$

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Return

$$(\pi_{\rho:w \rightarrow w'} \circ \text{return } x)(\eta) = [\text{id}_{w'}, (X\rho)x, \eta]$$

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Bind

Given $f : X \rightarrow TY$ in $\mathbf{V}$ and $\alpha \in TX$,

$$(\pi_{\rho:w \rightarrow w'} \alpha \gg f)(\eta') = [\rho'' \circ \rho', y, \eta''']$$

where

$$\text{cf. } \frac{\langle t, \eta' \rangle \Downarrow \langle a, \eta'' \rangle \quad \langle s[x \mapsto a], \eta'' \rangle \Downarrow \langle b, \eta''' \rangle}{\langle \text{let } x = t \text{ in } s, \eta' \rangle \Downarrow \langle b, \eta''' \rangle}$$

$$(\pi_{\rho} \alpha) \eta' = [\rho' : w' \rightarrow w'', x, \eta'']$$

$$(\pi_{\text{id}_{w''}} \circ f_{w''}(x))(\eta'') = [\rho'' : w'' \rightarrow w''', y, \eta''']$$

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State manipulation

$$\text{get}_c : \llbracket \mathbf{ref} \ c \rrbracket \rightarrow T \llbracket \mathbf{type} \ c \rrbracket$$

$$(\pi_\rho \circ \text{get}_c(\ell))(\eta) = [\text{id}_{w'}, \eta(\rho(\ell)), \eta]$$

$$\text{set}_c : \llbracket \mathbf{ref} \ c \rrbracket \times \llbracket \mathbf{type} \ c \rrbracket \rightarrow T\mathbb{1}$$

$$(\pi_\rho \circ \text{set}_c(\ell, a))(\eta) = [\text{id}_{w'}, \star, \eta[\rho(\ell) \mapsto \llbracket \mathbf{type} \ c \rrbracket \rho(a)]]$$

where $\eta(\ell) \stackrel{\text{def}}{=} \pi_\ell \eta$, and

$$\eta[\ell \mapsto x](\ell') = \begin{cases} x & \ell' = \ell \\ \eta(\ell') & \text{otherwise} \end{cases}$$

The full ground storage monad

Outline

For every $X \in \mathbf{V} = [\mathbb{W}, \mathbf{Set}]$:

$$TXw \subseteq \prod_{w \rightarrow w' \in \mathbb{W}} \mathbb{H}w' \Rightarrow \left(\sum_{w' \rightarrow w'' \in \mathbb{W}} Xw'' \times \mathbb{H}w'' \right) / \sim$$

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$$(TX)w = \int_{w \rightarrow w' \in w \downarrow u} \mathbb{H}w' \Rightarrow \left(\int_{w' \rightarrow w'' \in w' \downarrow u} X \circ uw'' \times \mathbb{H}w'' \right)$$

Allocation

Using

$$\begin{aligned} \partial_{w_0} : \mathbf{V} &\rightarrow \mathbf{V} & \text{init}_{w_0} : \prod_{(\ell:c) \in w_0} \partial_{w_0} [\mathbf{type} \ c] &\rightarrow \mathbb{E}(w, w \oplus w_0) \\ X &\mapsto X(- \oplus w_0) \end{aligned}$$

define

$$\begin{aligned} \text{new}_{w_0} : \prod_{(\ell:c) \in w_0} \partial_{w_0} [\mathbf{type} \ c] &\rightarrow T \prod_{(\ell:c) \in w_0} [\mathbf{ref} \ c] \\ (\pi_\rho \circ \text{new}_{w_0}(\langle a_\ell \rangle))\eta &= [\varepsilon^\star \rho, (\langle \varepsilon^\star \rho(\ell) \rangle)_{\ell \in w_0}, \mathbb{H}(\rho^\star \varepsilon)(\eta)] \end{aligned}$$

where $\varepsilon \stackrel{\text{def}}{=} \text{init} \langle a_\ell \rangle$.

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Garbage collection

Constant functors

Functors $X \in \mathbf{V}$ whose action $X\rho$ is a bijection.

Theorem (effect masking)

For every pair of constant functors $\Gamma, X \in \mathbf{V}$, every morphism $f : \Gamma \rightarrow TX$ factors uniquely through the monadic unit:

$$\begin{array}{ccc} \Gamma & \xrightarrow{f} & TX \\ & \searrow \text{runST } f & \nearrow \text{return}^T \\ & X & \end{array} \quad \begin{array}{c} \\ \\ = \end{array}$$

Interprets a multi-monadic-metalanguage with a **runST** construct [Launchbury-Peyton Jones'94]

Garbage collection

Proof.

$$\begin{array}{ll} \mathbb{1} \longrightarrow \mathbb{H} \multimap P(X \odot \mathbb{H}) & \text{in } \mathbf{V} \\ \hline \mathbb{H} \longrightarrow P(X \odot \mathbb{H}) & \text{in } \mathbf{C} \\ \hline \mathbb{H} \longrightarrow X \odot P\mathbb{H} & \text{by lemma} \\ \hline \mathbb{E}(\underline{0}, -) \longrightarrow X \odot P\mathbb{H} & \text{in } \mathbf{C} \\ \hline \mathbb{1} \longrightarrow (X \odot P\mathbb{H})\underline{0} & \text{in } \mathbf{Set}, \text{ by Yoneda} \\ \hline \mathbb{1} \longrightarrow X\underline{0} & \text{by lemma} \\ \hline \mathbb{1} \longrightarrow X & \text{in } \mathbf{V} \end{array}$$

and chase a generic morphism upwards.



A call-by-value calculus: λ_{ref} (denotational semantics)

With a standard denotational semantics we have:

Theorem (adequacy)

For all terms $\Gamma \vdash_w t_1, t_2 : \tau$, if $\llbracket t_1 \rrbracket = \llbracket t_2 \rrbracket$ then $\Gamma \vdash_w t_1 \simeq_{\text{ctx}} t_2 : \tau$.

Program transformations

The semantics validates the Hilbert-Post complete program equations for local ground state [Staton'10], which can be divided into three groups [*loc.cit.*, Melliés'14]:

- ▶ Global state equations [Plotkin-Power'02]
- ▶ Block algebra equations (allocation only)
- ▶ Local state equations (state-allocation interaction)

Conclusions

- ▶ A monad for full ground storage
- ▶ Heaps/stores functoriality over initialisations
- ▶ Decomposition into state-transformed encapsulation monad
- ▶ Evaluation: effect masking theorem and local ground state equations

Further work

- ▶ Haskell's **runST** and effect masking
- ▶ A denotational account of the value restriction [Wright'95]
- ▶ General storage
- ▶ Algebraic presentations, Hilbert-Post completeness, handlers
- ▶ Independence structures
- ▶ Combining effects
- ▶ Further analysis
- ▶ Garbage collectors and mallocators
- ▶ Comparison with ground case
- ▶ Nominal accounts