

Semantic foundations for type-driven probabilistic modelling

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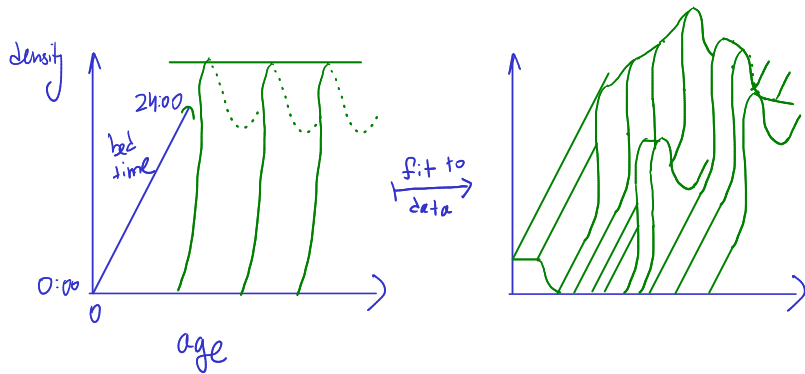


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Statistical & Probabilistic Modelling



ILLUSTRATIVE PURPOSES ONLY

Prob. Modelling as Programming

distributions describe Programs

⇓ [Saheb-Djafari '77
Kozen '78]

Programs describe distributions

Prob. Modelling as Programming

[Saheb-Djahromi '77]
[Kozen '78]

Programs describe distributions

Sophisticated samplers:

STLC + Sample + $\left\{ \begin{array}{l} \text{datatypes} \\ \text{recursion} \\ \text{polymorphism} \\ \dots \end{array} \right.$

Sophisticated Bayesian models: Sample + observe

likelihood/
density

modelling
via

PL features & stats components

Prior

inference

(generic/customized)

systems

(performance)

But Today:

Semantic foundations

Desiderata : Types

Discrete Spaces

finite spaces

Bool

$[n]$

success/failure
(Bernoulli)
coin tosses

week day
dice roll

infinite spaces

naturals

$\mathbb{N}$

↙ count

integers

$\mathbb{Z}$

↙ Priority

rational numbers

$\mathbb{Q}$

Desiderata : Types

Continuous Spaces

$\mathbb{R}$	$[a, b]$	$[0, \infty]$	$[0, 1]$
position	hour	scaling factors	probabilities

Discrete +

Desiderata : Types

Space Combinators

Products

$$\mathbb{R} \times \mathbb{R}$$

correlated
outcomes
(position + velocity)

Coproducts

$$\mathbb{R} \amalg [n]$$

coordinate or
location

Subspaces

$$S' \hookrightarrow \mathbb{R}^2$$

semantic
invariants

Quotients

$$\text{Bag } \mathbb{N}$$

point
processes

Discrete + Continuous

Desiderata: probability / measure

[Kolmogorov '33
equivalent
axioms]

Events $E \subseteq X$:

$$\{d \in [6] \mid \text{dice roll more than } 3: d \geq 3\}$$

Measure $\mu \Rightarrow$ Events $\xrightarrow{\text{Pr}} [0, \infty]$

$$E \mapsto \text{Pr}_{x \sim \mu} [x \in E]$$

measure
Outcomes
where E occurs

Desiderata: probability / measure

Measure $\mu \Rightarrow$ Events $\xrightarrow{Pr} [0, \infty]$

$$P_r[\emptyset] = 0$$

totality

$$P_r[E] = P_r[E \cap F] + P_r[E \cap F^c]$$

disjoint additivity

$$P_r\left[\bigcup_n E_n\right] = \sup_n P_r[E_n] \quad (E_n \subseteq E_{n+1})_n$$

(Scott) continuity

$$P_r[\text{whole space}] = 1$$

Probability

No-go #1:

Thm (Vitali, 1905) Assuming Axiom of Choice:

$\nexists P_r: \mathcal{P}\mathbb{R} \rightarrow [0, \infty]$ measure

$$P_r[E+a] = P_r[E]$$

translation
invariant

$$P_r[[a, b]] = b - a$$

measures
length

Borel subsets $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{P}(\mathbb{R})$

- Boolean subalgebra w.r.t. $\cap, \cup, \phi, \epsilon)^c$
- ω -chain closed:

$$\forall (E_n \subseteq E_{n+1})_n \in \mathcal{B}_{\mathbb{R}}^{\omega} \cdot \bigcup_n E_n \in \mathcal{B}_{\mathbb{R}}$$

- Every interval: $(a, b) \in \mathcal{B}_{\mathbb{R}}$

Thm: (Lebesgue 1902)

$\exists!$ $\mu: \mathcal{B}_{\mathbb{R}} \rightarrow [0, \infty]$ measure

translation
invariant

+

measures
length

Lebesgue Measure

Measurable space $A = (\underline{A}, \mathcal{B}_A)$

Events $\mathcal{B}_A \subseteq \mathcal{P}_A$

↖ set of points

- σ-field
- Boolean subalgebra w.r.t. $\cap, \cup, \phi, \complement$
 - ω-chain closed:

$$\forall (E_n \subseteq E_{n+1})_n \in \mathcal{B}_{\mathbb{R}}^{\omega} \cdot \bigcup_n E_n \in \mathcal{B}_{\mathbb{R}}$$

measurable: $A \xrightarrow{f} B$

$$f^{-1}[E] \in \mathcal{B}_A \iff E \in \mathcal{B}_B$$

classical theory [Kolmogorov 1933]

probability space

Ω
measurable
space

$\mu : \mathcal{B}_{\Omega} \rightarrow [0,1]$ Probability
measure

Modelling Foundation: Meas

discrete $(\mathbb{N}, P(\mathbb{N}))$

+ their measures

continuous $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$

Combinators:

products, subspaces

(limit
structure)

coproducts quotients

(colimit
structure)

No-Go as PL Semantics:

Thm [Aumann'61] Let $\mathbb{R}^{\mathbb{R}} := \text{Meas}(\mathbb{R}, \mathbb{R})$

$\nexists B_0 \subseteq \mathcal{P}(\mathbb{R}^{\mathbb{R}})$ σ -field

eval: $(\mathbb{R}^{\mathbb{R}}, B_0) \times \mathbb{R} \xrightarrow{(f, x) \mapsto fx} \mathbb{R}$ measurable

Consequences

$\Rightarrow$ no-go for event space $B_{(B_{\mathbb{R}})}$

$\Rightarrow$ no-go for Π -types

$\Rightarrow$ no-go for:

- Functional programming
- OO Programming ...

Semantic Tradition: Probabilistic Powerdomains

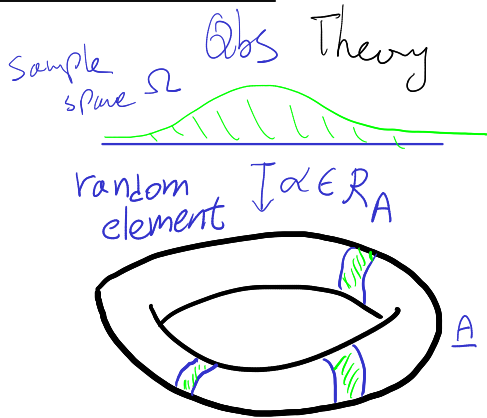
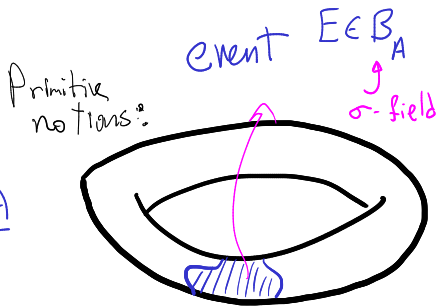
[Plotkin '82, Graham '87, Jones-Plotkin '89, Tix, Jung 97
Tix 98, Mislove '00+... Keimel '03,
J. Goubault-Lorrecq '07 +... + apologies]

Recent developments

- Boolean-valued models [Bacci et al.'18]
- Probabilistic coherence spaces with cones [Ehrhard et al.'18]
- Measurable event structures [Paquet-Winskel'18] [de Amorim et al.'22]
- Banach spaces with structure [Dalqvist, Kozen'20]
- Interval domains [Jia et al.'21a+b, Di Gianantonio, Eldorot'24]
- Random probability Sheaf Toposes [Simpson]

Quasi-Borel Spaces [Staton et al. '17] cf. [Forné '21]

Measure Theory



Derived

random element

$(\alpha: \Omega \rightarrow A) \in \mathcal{R}_A$

metaphorology

event

$E \in \mathcal{B}_A$

Weak de Finetti Thm

Applications

Verify Bayesian inference implementations

Design modelling languages & inference systems
+ implementations

Probabilistic Networking PL

Formalise

semantics for expected cost analysis

Also:

semantics for name generation

Weak de Finetti Thm [Staton et al.'17] Applications

Verify Bayesian inference implementations [Scribner et al.'18]

Design modelling languages & inference systems

+ implementations: MonalBayes [Scribner et al.'18, now Tweag.io]

Lazy PPL [Staton et al.'23] (parts of) Gen [Lew et al.'20]

Domain Theory [Vakár et al.'19]

Probabilistic Networking PL [Vandenbroucke, Schrijvers'20]

Formalise [Hirata et al.'19 + '23]

Semantics for expected cost analysis [de Amorim'24]

Also:

semantics for name generation [Sabok et al.'21]

Rest of talk : mini-tutorial via types

- 1) Metaphorologies & Qbs
- 2) Simple-type structure
- 3) Standard spaces



Course

PhD
Programs



CDT:
Dependable AI



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Theme: Working with quasi Borel spaces:

1) types for abstracting over
measurability

2) types for organising probabilistic
concepts

Not today:

- I) Colimit structure & Probability spaces
- II) dependently-typed structure
- III) random variable spaces & Stochastic Processes



See
course



Def: Metaphorology over X Set
"random elements" $\mathcal{R}$ "points"

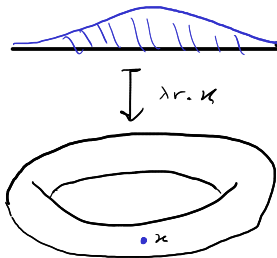
$\mathcal{R} \subseteq X^{\mathbb{R}}$ $\models$ 3 Closure axioms:

- Constants:

$$\frac{x \in X}{\underline{x} := (\lambda r. x) \in \mathcal{R}}$$

- Precomposition:

- recombination



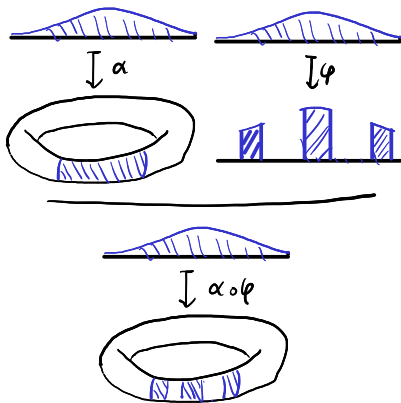
Def: Metaphorology over X Set
 & "random elements"
 & "points"

$\mathcal{R} \subseteq X^{\mathbb{R}} \models 3 \text{ Closure axioms:}$

- Precomposition:

$\alpha \in \mathcal{R}_X \quad \varphi: \mathbb{R} \rightarrow \mathbb{R} \quad \text{Borel measurable}$

$\varphi \circ \alpha: \mathbb{R} \xrightarrow{\varphi} \mathbb{R} \xrightarrow{\alpha} X, \in \mathcal{R}$



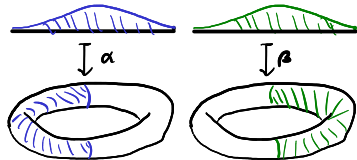
Def: Metaphorology over X Set
 $\mathcal{R}$ "points"

$\mathcal{R} \subseteq X^{\mathbb{R}} \models 3 \text{ Closure axioms:}$

- re combination

$$\vec{\alpha} \in \mathcal{R}_X^{\mathbb{N}} \quad \vec{E} \in \mathcal{B}_{\mathcal{R}}^{\mathbb{N}} \quad \mathbb{R} = \bigoplus_{n=0}^{\infty} E_n$$

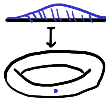
$$[E_n, \alpha_n]_n := \lambda r. \left\{ \begin{array}{l} r \in A_n: \alpha_n r \in \mathcal{R} \\ \vdots \end{array} \right.$$

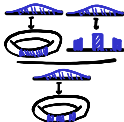


Def: Quasi-Borel space $X = (LX, R_X)$

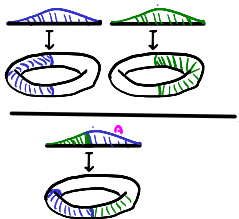
Set + metaphorology

$R_X \subseteq LX^{LR_X}$ closed under:

- Constant S : 

- Precomposition: 

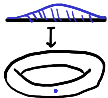
- recombination

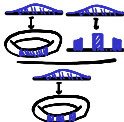


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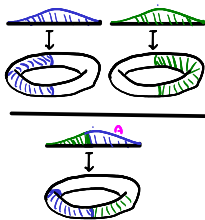
- precomposition: 

Topos Theorists:

Concrete sheaves over

$(Sbs, \text{Countable covers})$

- recombination



Examples

- $\mathbb{R} = (\mathbb{R}, \text{Meas}(\mathbb{R}, \mathbb{R}))$ qbs underlying $\mathbb{R}$

- Generalise:

A meas space

$\underline{A}_{\text{qbs}} := (\underline{A}, \text{Meas}(\mathbb{R}, A))$ qbs

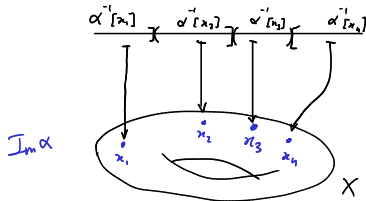
Examples

recombination of constants

Def: $\alpha: \mathbb{R} \rightarrow X$ σ -Simple: $\alpha = [E_n, x_n]_{n \in I}$

$$I \hookrightarrow \mathbb{N} \quad \vec{E} \in \mathcal{B}^I$$

$$\vec{x} \in X^I \quad \mathbb{R} = \bigcup_n E_n$$



discrete qbs on X

X sets

$$\mathcal{X}^{\text{qbs}} := (X, \sigma\text{-simple}(\mathbb{R}, X)) \text{ qbs}$$

Examples

Indiscrete qbs on X

X set

$$\underbrace{X}_{\text{Obs}} := (X, X^{\mathbb{R}})$$

↪ all functions

Def: Obs morphism $f: X \rightarrow Y$

- function $f: X_1 \rightarrow Y_1$

$$- \forall \begin{array}{c} \mathbb{R} \\ \alpha \downarrow \\ X_1 \end{array} \in R_X \quad , \quad \begin{array}{c} \mathbb{R} \\ \alpha \downarrow \\ X_1 \\ f \downarrow \\ Y_1 \end{array} \in R_Y$$

Example

- Constant Functions:

Since:

f constant $\Rightarrow f \circ \alpha$ constant

Example Extending Meas

A, B meas spaces $f: A \rightarrow B$ in Meas

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \text{abs} & & \downarrow \text{abs} \end{array}$$

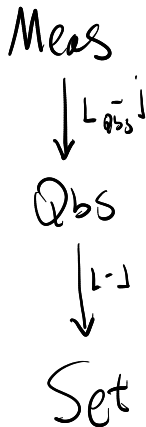
Example Category Obs

Identities:

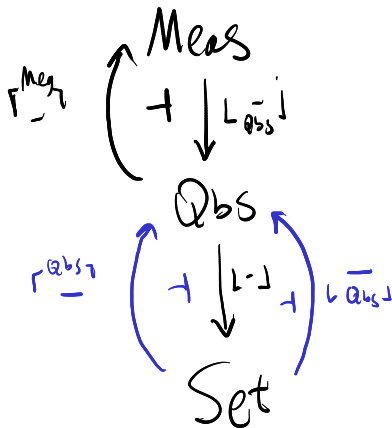
$$ib_A := (\lambda x. x): A \rightarrow A$$

Composition:

$$\begin{array}{ccc} A & \xrightarrow{\lambda x. g(f x)} & C \\ & f \searrow \quad \nearrow g & \\ & B & \end{array}$$



generates limits & colimits
↳ (lifts + Preserves)



NB, for C.B. not cohesive

Rest of talk : mini-tutorial via types

1) Metaphorologies & Qbs

2) Simple-type structure

3) Standard spaces



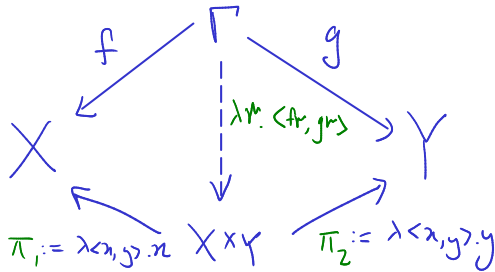
Course

Product $(X \times Y, \pi_1, \pi_2)$:

- $L_{X \times Y} = L_X \times L_Y$ *necessarily!*

- $R_{X \times Y} = \{ \lambda r. (\alpha r, \beta r) \mid \alpha \in R_X, \beta \in R_Y \}$

correlated
random
elements



$E_{\mathbb{R}}$

$\mathbb{R}^n, \mathbb{R}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}$

$(+), (\cdot) : \mathbb{R}^2 \rightarrow \mathbb{R}$

inf, sup

$\liminf, \limsup : \overline{\mathbb{R}^{\mathbb{N}}} \rightarrow \overline{\mathbb{R}}$

$Qbs \models 1^{st}$ order simple-type theory (like meas)

$$\frac{(x:A) \in \Gamma}{\Gamma \vdash x:A} \text{var} \qquad \frac{\Gamma \vdash M:A \quad f:A \rightarrow B \text{ in } Qbs}{\Gamma \vdash \llbracket f \rrbracket M : B} \text{reflect}$$

$$\frac{\Gamma \vdash M_i : A_i}{\Gamma \vdash \langle M_1, \dots, M_n \rangle : A_1 \times \dots \times A_n} \text{type} \qquad \frac{\Gamma \vdash M : A_1 \times \dots \times A_n \quad \Gamma, (x_i:A_i) \vdash K : B}{\Gamma \vdash \text{let}(x_1, \dots, x_n) = M \text{ in } K : B} \text{let}$$

smooth internalisation/externalisation

Function Spaces

Straightforward!

$$- \mathcal{Y}^X := \text{Obs}(X, \mathcal{Y})$$

$$- \mathcal{R}_{\mathcal{Y}^X} := \text{curry}[\text{Obs}(\mathbb{R} \times X, \mathcal{Y})]$$

$$= \left\{ \alpha: \mathbb{R} \rightarrow \mathcal{Y}^X \mid \lambda(r, x). \alpha r x: \mathbb{R} \times X \rightarrow \mathcal{Y} \right\}$$

$$- \text{eval}: \mathcal{Y}^X \times X \rightarrow \mathcal{Y}$$

$$\text{eval}(f, x) := fx$$

Simple type theory

Obs $\models$ STLC

$$\frac{\Gamma, x:A \vdash M:B}{\Gamma \vdash \lambda x:A. M : B^A}$$

$$\frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash P:A}{\Gamma \vdash MP : B}$$

Ex.

$$\begin{aligned} \limsup &: \overline{\mathbb{R}}^{N \times \mathbb{R}} \longrightarrow \overline{\mathbb{R}}^{\mathbb{R}} \\ \limsup \vec{f} &:= \lambda r. \limsup_{n \rightarrow \infty} f_n r \end{aligned}$$

Random element space

$$R_X := X^{\mathbb{R}} \quad \text{since}$$

$$\llbracket X^{\mathbb{R}} \rrbracket = R_X \text{ as sets.}$$

Random element Space

$$R_X := X^{\mathbb{R}} \quad \text{since} \quad \llbracket X^{\mathbb{R}} \rrbracket = R_X \text{ as sets.}$$

Why?

$$(\subseteq) \quad \alpha \in \llbracket X \rrbracket^{\mathbb{R}} \Rightarrow \alpha: \mathbb{R} \rightarrow X \text{ in Obs.}$$

$$\text{id}_{\mathbb{R}}: \mathbb{R} \rightarrow \mathbb{R} \text{ measurable} \Rightarrow \text{id} \in R_{\mathbb{R}}$$

$$\Rightarrow \alpha = \alpha \circ \text{id} \in R_X$$

$$(\supseteq) \quad \alpha \in R_X \Rightarrow \forall \psi \in R_{\mathbb{R}} = \text{Meas}(\mathbb{R}, \mathbb{R}). \quad \alpha \circ \psi \in R_X \Rightarrow \alpha: \mathbb{R} \rightarrow X \Rightarrow \alpha \in \llbracket X \rrbracket^{\mathbb{R}}$$

Pre composition
↙

Random element Space

Ex: Riemann-Stieltjes integrals & Lebesgue-Stieltjes measures

$$\underline{R}X := \underline{R}_X$$

$$\begin{matrix} \vee \\ \int \end{matrix} X \rightarrow \underline{R}X$$

$$\int_x := \underline{x}$$

$$\int: \underline{R}X \times (\underline{R}Y)^X \rightarrow \underline{R}Y$$

$$\int: \underline{R}X \times [0, \infty]^X \rightarrow [0, \infty]$$

$$\int d\alpha \varphi := \int dN(\varphi \circ \alpha)$$

$\hookrightarrow \mathbb{R} \rightarrow [0, \infty]$

$$\begin{matrix} \vee \\ \int \end{matrix} d\alpha \varphi$$

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\cong} & \mathbb{R} \times \mathbb{R} \xrightarrow{\alpha \times id} X \times \mathbb{R} \\ \downarrow := & & \downarrow f \times id \\ Y & \xleftarrow{eval} & Y \times \mathbb{R} \end{array}$$

Subspaces

A qbs

$X \subseteq A$ set

$$\text{cast} : \{x \mid x \in X\} \hookrightarrow A$$

Subspace

$$\mathcal{R}_{\{X\}} := \{ \alpha : \mathbb{R} \rightarrow X \mid \alpha \in \mathcal{R}_A \} \quad \text{cast } \alpha := \alpha$$

$$\text{Ex: 1) } S' := \{ \vec{x} \in \mathbb{R}^2 \mid \|\vec{x}\| = 1 \} \hookrightarrow \mathbb{R}^2$$

2) Skorokhod Representation:

$$\text{cdf: } \underline{\mathbb{R}} \overline{\mathbb{R}} \xrightarrow{\sim} \{ f : \overline{\mathbb{R}} \rightarrow \mathbb{R} \mid f(-\infty) = 0, f(\infty) = 1, \\ f \text{ càdlàg, monotone} \} \hookrightarrow \mathbb{R}^{\overline{\mathbb{R}}}$$

Events

$$\mathcal{B}_A := \left\{ A \subseteq \mathcal{A} \mid \forall \alpha \in \mathcal{R}_x. \alpha \Vdash [A] \in \mathcal{B}_{\mathcal{R}} \right\} \quad \sigma\text{-field}$$

$$\mathcal{B}_A \cong \text{Obs}(A, [\text{Bool}]) \Rightarrow \mathcal{B}_A \cong \text{Bool}^A$$

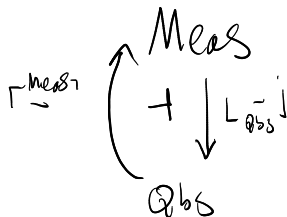
N.B. Topos Theorists: $[\text{Bool}]$ classifies events via p.b.

Descriptive

set theorists: $\mathcal{B}_{\mathcal{B}_{\mathcal{R}}}$ are the Borel-on-Borel sets from
cf. [Sabou et al. '21]

A qbs

$\vdash^{Meas} A := (A, B_A)$ Meas space



Meas vs Obs

By generalities: σ -field on $\text{Meas}(\mathbb{R}, \mathbb{R})$

$$\begin{array}{c} \text{Meas} \\ \mathbb{R} \end{array} \rightarrow \begin{array}{c} \text{Meas} \\ \mathbb{R} \end{array} \times \mathbb{R} \xrightarrow{\text{red X}} \begin{array}{c} \text{Meas} \\ \mathbb{R} \end{array} = \mathbb{R}$$

No factorisation
by
Aumann's
Theorem.

$$\left(\begin{array}{c} \text{So} \\ \text{eval} \end{array} \right) \begin{array}{c} \text{Meas} \\ \mathbb{R} \end{array} \times \mathbb{R} \neq \begin{array}{c} \text{Meas} \\ \mathbb{R} \end{array} \times \begin{array}{c} \text{Meas} \\ \mathbb{R} \end{array}$$

Rest of talk : mini-tutorial via types

- 1) Metaphorologies & Qbs
- 2) Simple-type structure
- 3) Standard spaces



Course

Borel Embedding:

$$e: A \hookrightarrow B$$

When:

$$\lambda x. ex: A \xrightarrow{\lambda} e[A] \hookrightarrow B \quad \& \quad e[A] \in \mathcal{B}_B$$

$$\underline{En}: \mathcal{N}_{\text{ull}} := \left\{ E \text{ Borel null-set} \right\} \hookrightarrow \mathcal{B}_{\mathbb{R}}$$

characterised by: $\lambda E. \left(\int d\mu [-eE] \stackrel{?}{=} 0 \right)$

Non-examples ~ [Sabok et al. '21]

$$- \{ A \in \mathcal{B}_{\mathbb{R}} \mid A \neq \emptyset \} \hookrightarrow \mathcal{B}_{\mathbb{R}}$$

$$- \{ (A_1, A_2) \in \mathcal{B}_{\mathbb{R}}^2 \mid A \subseteq B \} \hookrightarrow \mathcal{B}_{\mathbb{R}}^2$$

$$- \{ A \in \mathcal{B}_{\mathbb{R}} \mid A \text{ open} \} \hookrightarrow \mathcal{B}_{\mathbb{R}}$$

Standard Borel Spaces

Def: A qbs S is **standard Borel**

$$S \hookrightarrow \mathbb{R}$$

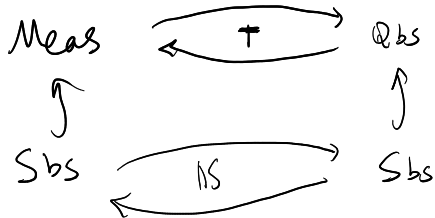
Slogan:

Standard $\equiv$ concrete $\equiv$ well behaved

Standard Borel Spaces

Def: A qbs S is **standard Borel**

$$S \hookrightarrow \mathbb{R}$$



Slogan: Qbs conservative extension of Sbs

Classical measure theory

- 1) Define σ -field
- 2) Prove it standard
- 3) show relevant operations measurable

With Qbs

- 1) use type formers
 $\rightsquigarrow$ support operations
 by construction
- 2) Prove it standard
- 3) characterise events

Uniform convergence space

Space of continuous functions:

$$1) \quad C_0 := \{f: \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ continuous}\} \hookrightarrow \mathbb{R}^{\mathbb{R}}$$

$$\text{eval}: C_0 \times \mathbb{R} \rightarrow \mathbb{R}$$

$$\text{eval} \langle f, x \rangle := (\text{cast } f) x$$

$$\Gamma \vdash f : \mathbb{R} \rightarrow \mathbb{R} \qquad \forall m. \llbracket f \rrbracket m \text{ cts}$$

$$\Gamma \vdash (f, \text{cts}) : C_0$$

Uniform convergence space

2) Prove it standard:

Code $\hookrightarrow \mathbb{R}^{\mathbb{Q}}$

$$:= \left\{ \vec{y} \in \mathbb{R}^{\mathbb{Q}} \mid \begin{array}{l} \forall a, b \in \mathbb{Q}, \varepsilon \in \mathbb{Q}^+ \\ \exists \delta \in \mathbb{Q}^+ \forall p, q \in \mathbb{Q}^+ \cap [a, b] \left\{ \begin{array}{l} |p - q| < \delta \Rightarrow |y_p - y_q| < \varepsilon \end{array} \right. \end{array} \right\}$$

idea:

f cts on $\mathbb{R} \Leftrightarrow$

$f|_{[a,b]}$ uniformly cts

Uniform convergence space

2) Prove it standard:

$$\begin{array}{c} \text{--- } \lambda q := \lambda f. \lambda g. f q \\ C_0 \xrightarrow{\quad \approx \quad} C_{\text{ode}} \\ \text{--- } \lambda \vec{y}, \lambda x. \\ \text{let } \vec{q} : \mathbb{Q}^n \\ \text{= rational Approx } x \\ \text{in } \lim_{n \rightarrow \infty} y_{q_n} \end{array}$$

idea:

f cts on $\mathbb{R} \Leftrightarrow$
 $f|_{[a,b]}$ uniformly cts

Characterising B_A

$d: \underline{A} \times \underline{A} \rightarrow [0, \infty]$ metric A gbs

d compatible: $d: A \times A \rightarrow [0, \infty]$ measurable

d has measurable limits:

$\text{converges}_d := \{ \vec{a} \text{ converges} \} \hookrightarrow A^{\mathbb{N}}$

and $\lim: \text{converges}_d \rightarrow A$ measurable.

Thm: A -compatible w/ measurable limits d .
 d separable $\Rightarrow \sigma[\mathcal{G}_d] = \mathcal{B}_A$

Thm: A -compatible w/ measurable limits d .

$$d \text{ separable} \Rightarrow \sigma[\mathcal{U}_d] = \mathcal{B}_A$$

Application:

$$3) \mathcal{B}_{C_0} = \sigma(\text{Uniform topology}) = \sigma[\mathcal{U}_{d_U}] :$$

Proof:

$$d_U(f, g) := \sup_{r \in \mathbb{R}} |f_r - g_r| = \sup_{r \in \mathbb{Q}} |f_r - g_r| \text{ meas.}$$

$$\text{Converge}_{d_U} = \text{Cauchy}_{d_U} \Leftrightarrow \exists A < \infty \forall \epsilon > 0 \exists n \forall i, j \geq n \quad d_U(f_i, f_j) < \epsilon$$

$$\lim \vec{f} := \lim_{n \rightarrow \infty} f_n$$

d separable via Weierstrass's Approx. Thm + Rational Bernstein Polynomials

Defn for the Skorokhod space:

$$D[a,b) := \left\{ f: [a,b) \rightarrow \mathbb{R} \mid f \text{ càdlàg} \right\} \hookrightarrow \mathbb{R}^{[a,b)}$$

Wip: Effros quasi-Borel space

A Polish

$$\mathcal{E}_A := \{ F \in \mathcal{B}_A \mid F \text{ closed} \} \hookrightarrow \mathcal{B}_A$$

$\rightsquigarrow \mathcal{E}_{l_2} =: \mathcal{P}_0$ Tarski universe
for Polish spaces

Concluding remarks: mini-tutorial via types



1) Metaphorologies & Qbs

2) Simple-type structure

3) Standard spaces



Talk to me:

I) Colimit structure & Probability spaces PhD

II) dependently-typed structure



III) random variable spaces
& Stochastic

Processes

Course