A domain theory for quasi-Borel spaces

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Statistical probabilistic programming

$$[\![-]\!] : \text{programs} \to \text{distributions}$$

- Continuous types: $\mathbb{R}, [0, \infty]$
- Probabilistic effects:

normally distributed sample

$$\mathbf{sample}(\mu, \sigma) : \mathbb{R}$$

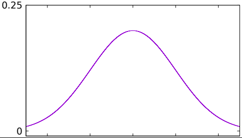
scale distribution by r

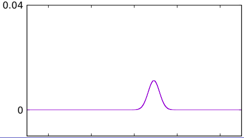
$$\frac{r : [0, \infty]}{\mathbf{score}(r) : \mathbf{1}}$$

conditioning/fitting to observed data

$$[\![\mathbf{sample}(0, 2)]\!]$$

$$\left[\!\!\left[\begin{array}{l}\mathbf{let}\ x{=}\mathbf{sample}(0{,}2)\\ \mathbf{in}\ \mathbf{score}(normalPdf(1.1|\ x{,}1));\\ \quad \mathbf{score}(normalPdf(1.9|2x{,}1));\\ \quad \mathbf{score}(normalPdf(2.7|3x{,}1)); x\end{array}\right]\!\!\right]$$





Statistical probabilistic programming

- Commutativity/exchangability

$$\left[\!\!\left[\begin{array}{l}\mathbf{let}\, x = M\ \mathbf{in}\\ \mathbf{let}\, y = N\ \mathbf{in}\\ f(x,y)\end{array}\right]\!\!\right] = \left[\!\!\left[\begin{array}{l}\mathbf{let}\, y = N\ \mathbf{in}\\ \mathbf{let}\, x = M\ \mathbf{in}\\ f(x,y)\end{array}\right]\!\!\right]$$

Exact Bayesian inference using disintegration [Shan-Ramsey'17]

Fubini's:

$$\int [\![M]\!](\mathrm{d}x) \int [\![N]\!](\mathrm{d}y) f(x,y) = \int [\![N]\!](\mathrm{d}y) \int [\![M]\!](\mathrm{d}x) f(x,y)$$

probabilitity distributions ✔

σ-finite distributions ✔

arbitrary distributions ✗

s-finite distributions ✔

not closed under push-forward

full definability [Staton'17]

Statistical probabilistic programming

Express continuous distributions with:

- Higher-order functions

measure theory ✗

measurable cones and stable measurable functions ✓

quasi-Borel spaces ✓

[Heunen et al.'17]

[Ehrhard-Pagani-Tasson'18]

Theorem (Aumann'61)

No σ-algebra over $\mathbf{Meas}(\mathbb{R}, \mathbb{R})$ with measurable evaluation:

$$\mathrm{eval} : \mathbf{Meas}(\mathbb{R}, \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$$

modular implementation of Bayesian inference algorithms [Ścibior et al.'18a+b]

- Inductive types and bounded iteration
- Term recursion
- Type recursion

domain theory [this work]

Iso-recursive types: FPC

type variable contexts $\Delta = \{\alpha_1, \ldots, \alpha_n\}$

[Fiore-Plotkin'94]

$$\frac{\Delta, \alpha \vdash_{\mathrm{k}} \tau : \mathsf{type}}{\Delta \vdash_{\mathrm{k}} \mu\alpha.\tau : \mathsf{type}}$$

type recursion

$$\begin{aligned} Lam = \ \mu\alpha.\{&\mathrm{Bool}\{\mathrm{True} \mid \mathrm{False}\} \\ &\mid \mathrm{App}(\alpha * \alpha) \\ &\mid \mathrm{Abs}(\alpha \to \alpha)\} \end{aligned}$$

$\tau = \mu\alpha.\sigma$

$$\frac{\Gamma \vdash t : \sigma[\alpha \mapsto \tau]}{\Gamma \vdash \tau.\mathbf{roll}\,(t) : \tau} \qquad \frac{\Gamma \vdash t : \tau \qquad \Gamma, x : \sigma[\alpha \mapsto \tau] \vdash s : \rho}{\Gamma \vdash \mathbf{match}\, t\, \mathbf{with}\, \mathbf{roll}\, x \Rightarrow s : \rho}$$

Iso-recursive types: FPC

[Fiore-Plotkin'94]

type variable contexts
$\Delta = \{\alpha_1, \ldots, \alpha_n\}$

$$\frac{\Delta, \alpha \vdash_{\mathrm{k}} \tau : \mathsf{type}}{\Delta \vdash_{\mathrm{k}} \mu\alpha.\tau : \mathsf{type}}$$

type recursion

$\omega\mathbf{Cpo}$-enriched category of domains

$$[\![\Delta \vdash_{\mathrm{k}} \tau : \mathsf{type}]\!] : (\mathcal{C}^{\mathrm{op}})^n \times \mathcal{C}^n \to \mathcal{C}$$

locally continuous functor

$$[\![\Delta \vdash_{\mathrm{k}} \mu\alpha.\tau : \mathsf{type}]\!] = \text{minimal invariants}$$

[Freyd'91,92, Pitts'96]

Challenge

- probabilistic powerdomain
- commutativity/Fubini
- domain theory
- higher-order functions

continuous domains
[Jones-Plotkin'89]

open problem
[Jung-Tix'98]

traditional approach:

domain $\mapsto$ Scott-open sets $\mapsto$ Borel sets $\mapsto$ distributions/valuations

our approach:

following
[Ehrhard-Pagani-Tasson'18]

$$(\text{domain}, \text{quasi-Borel space}) \mapsto \text{distributions}$$

separate
but compatible

- $\omega\mathbf{Qbs}$: a category of pre-domain quasi-Borel spaces
- M: commutative probabilistic powerdomain over $\omega\mathbf{Qbs}$

Theorem (adequacy)

M adequately interprets:

- *Statistical FPC*
- *Untyped Statistical λ-calculus*

Plan

- $\omega\mathbf{Qbs}$
- a powerdomain over $\omega\mathbf{Qbs}$
- a domain theory for $\omega\mathbf{Qbs}$

Quasi-Borel pre-domains

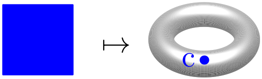
ω-qbs:

set — partial order on $\underline{X}$ — subset of functions $\mathbb{R} \to \underline{X}$

$$X = (\underline{X}, \leq_X, M_X)$$

ω-cpo — quasi-Borel space

- $\lambda\_.x \in M_X$



s.t.:

pointwise ω-chain — pointwise lub

$$(\alpha_n) \in M_X^\omega \implies \bigvee_n \alpha_n \in M_X$$

Morphisms $f : X \to Y$: Scott continuous qbs maps

monotone and $f \bigvee_n x_n = \bigvee_n f x_n$

$\forall \alpha \in M_X.$ $f \circ \alpha \in M_Y$

Quasi-Borel pre-domains

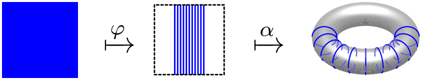
ω-qbs:

$$X = (\underline{X}, \leq_X, M_X)$$

set → $\underline{X}$; partial order on $\underline{X}$ → $\leq_X$; subset of functions $\mathbb{R} \to \underline{X}$ → M_X

ω-cpo: $(\underline{X}, \leq_X)$; quasi-Borel space: $(\underline{X}, M_X)$

- $\lambda\_.x \in M_X$
- $\alpha \in M_X \implies \alpha \circ \varphi \in M_X$ (where $\mathbb{R} \xrightarrow{\varphi}_{\text{Borel}} \mathbb{R}$)



s.t.:

$$(\alpha_n) \in M_X^\omega \implies \bigvee_n \alpha_n \in M_X$$

pointwise ω-chain: $(\alpha_n) \in M_X^\omega$; pointwise lub: $\bigvee_n \alpha_n$

Morphisms $f : X \to Y$: Scott continuous qbs maps

continuous: monotone and $f \bigvee_n x_n = \bigvee_n f\, x_n$

qbs maps: $\forall \alpha \in M_X.\ f \circ \alpha \in M_Y$

Quasi-Borel pre-domains

ω-qbs:

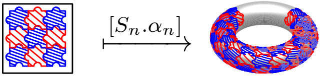
set; partial order on $\underline{X}$; subset of functions $\mathbb{R} \to \underline{X}$

$$X = (\underline{X}, \leq_X, M_X)$$

ω-cpo; quasi-Borel space

$\mathbb{R} \xrightarrow{\varphi}_{\text{Borel}} \mathbb{R}$

- $\lambda\_.x \in M_X$
- $\alpha \in M_X \implies \alpha \circ \varphi \in M_X$
- $(\alpha_n \in M_X)_{n \in \mathbb{N}} \implies [r \in S_n.\alpha(r)] \in M_X$



Borel measurable countable partition $\mathbb{R} = \biguplus_{n\in\mathbb{N}} S_n$

s.t.:

pointwise ω-chain; pointwise lub

$$(\alpha_n) \in M_X^\omega \implies \bigvee_n \alpha_n \in M_X$$

Morphisms $f : X \to Y$: Scott continuous qbs maps

monotone and $f \bigvee_n x_n = \bigvee_n f\, x_n$

$\forall \alpha \in M_X.$ $f \circ \alpha \in M_Y$

Quasi-Borel pre-domains

$$X = (\underline{X}, \leq_X, M_X)$$

- $\lambda\_.x \in M_X$
- $\alpha \in M_X \implies \alpha \circ \varphi \in M_X$
- $(\alpha_n \in M_X)_{n \in \mathbb{N}} \implies [r \in S_n.\alpha(r)] \in M_X$

s.t.:

$$(\alpha_n) \in M_X^{\omega} \qquad \implies \qquad \bigvee_n \alpha_n \in M_X$$

Example

$S = (\underline{S}, \Sigma_S)$ measurable space

$$\left(\underline{S}, =, \{\alpha : \mathbb{R} \to \underline{S} | \alpha \text{ Borel measurable}\}\right)$$

so $\mathbb{R} \in \omega\mathbf{Qbs}$

Quasi-Borel pre-domains

$$X = (\underline{X}, \leq_X, M_X)$$

- $\lambda\_.x \in M_X$
- $\alpha \in M_X \implies \alpha \circ \varphi \in M_X$
- $(\alpha_n \in M_X)_{n\in\mathbb{N}} \implies [r \in S_n.\alpha(r)] \in M_X$

s.t.:

$$(\alpha_n) \in M_X^\omega \qquad \implies \qquad \bigvee_n \alpha_n \in M_X$$

Example

$P = (\underline{P}, \leq_P)$ ω-cpo

$$\left(\underline{P}, \leq_P, \left\{ \bigvee_k [\_ \in S_n^k.a_n^k] \middle| \forall k.\mathbb{R} = \biguplus_n S_n^k \right\}\right)$$

so $\mathbb{L} = \left([0,\infty], \leq, \left\{\alpha : \mathbb{R} \to [0,\infty] \middle| \alpha \text{ Borel measurable}\right\}\right) \in \omega\mathbf{Qbs}$

Quasi-Borel pre-domains

$$X = (\underline{X}, \leq_X, M_X)$$

- $\lambda\_.x \in M_X$
- $\alpha \in M_X \implies \alpha \circ \varphi \in M_X$
- $(\alpha_n \in M_X)_{n\in\mathbb{N}} \implies [r \in S_n.\alpha(r)] \in M_X$

s.t.:

$$(\alpha_n) \in M_X^\omega \qquad \implies \qquad \bigvee_n \alpha_n \in M_X$$

Example

X ω-qbs

$$X_\perp := \left(\{\perp\} + \underline{X}, \perp \leq \underline{X}, \left\{[S.\perp, S^{\complement}.\alpha] \middle| \alpha \in M_X, S \text{ Borel}\right\}\right)$$

Quasi-Borel pre-domains

Theorem

$\omega\mathbf{Qbs} \to \omega\mathbf{Cpo} \times \mathbf{Qbs}$ *creates limits*

Products

$$\underline{X_1 \times X_2} = \underline{X}_1 \times \underline{X}_2 \qquad x \le y \iff \forall i. x_i \le y_i$$

$$M_{X_1 \times X_2} = \left\{ (\alpha_1, \alpha_2) : \mathbb{R} \to \underline{X}_1 \times \underline{X}_2 \middle| \forall i. \alpha_i \in M_{X_i} \right\}$$

correlated random elements

Exponentials

- $\underline{Y^X} = \{ f : \underline{X} \to \underline{Y} | f \text{ Scott continuous qbs morphism} \}$
  $= \mathbf{Qbs}(X, Y)$
- $f \le g \iff \forall x \in \underline{X}. f(x) \le g(x)$
- $M_{Y^X} = \left\{ \alpha : \mathbb{R} \to \underline{Y^X} \middle| \begin{array}{l} \text{uncurry}\, \alpha : \mathbb{R} \times X \to Y \\ \text{Scott continuous qbs morphism} \end{array} \right\}$

so $\underline{Y}^{\mathbb{R}} = M_Y$

Characterising $\omega\mathbf{Qbs}$

$$\mathbf{Sbs} \xhookrightarrow{\text{Yoneda}} [\mathbf{Sbs}^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{cpp}}$$

$$\mathbf{Sbs} \xhookrightarrow{\text{Yoneda}} \text{SepSh} \hookrightarrow [\mathbf{Sbs}^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{cpp}} \quad (=)$$

countable product preserving [Staton et al.'16]

$F : \mathbf{Sbs}^{\mathrm{op}} \to \mathbf{Set}$ separated: $F\mathbb{R} \xrightarrow{\left(F(\mathbb{R} \xleftarrow{r} \mathbb{1})\right)_{r\in\mathbb{R}}} (F\mathbb{1})^{\mathbb{R}}$ injective

Thm: $\mathbf{Qbs} \simeq$ SepSh

[Heunen et al.'17]

$$\mathbf{Sbs} \xhookrightarrow{\text{Yoneda}} [\mathbf{Sbs}^{\mathrm{op}}, \omega\mathbf{Cpo}]_{\mathrm{cpp}}$$

$$\mathbf{Sbs} \xhookrightarrow{\text{Yoneda}} \omega\text{SepSh} \hookrightarrow [\mathbf{Sbs}^{\mathrm{op}}, \omega\mathbf{Cpo}]_{\mathrm{cpp}} \quad (=)$$

$f(x) \le f(y) \implies x \le y$

$F : \mathbf{Sbs}^{\mathrm{op}} \to \omega\mathbf{Cpo}$ ω-separated: $F\mathbb{R} \xrightarrow{\left(F(\mathbb{R} \xleftarrow{r} \mathbb{1})\right)_{r\in\mathbb{R}}} (F\mathbb{1})^{\mathbb{R}}$ full

Thm: $\omega\mathbf{Qbs} \simeq_{\omega\mathbf{Cpo}} \omega$SepSh

Characterising $\omega\mathbf{Qbs}$

Grothendieck quasi-topos $\mathbf{Qbs}$

strong subobject classifier:

$$\underline{\Omega} = 2 \qquad M_\Omega = 2^{\mathbb{R}}$$

strong monos:

$X \overset{f}{\rightarrowtail} Y$

$(f\circ)^{-1}[M_Y] = M_X$

$P^2 \xrightarrow{\leq_P} \Omega$

qbs

ω-chain$(P) \xrightarrow{\bigvee} P$

Internal ω-cpo P:

$$(\underline{P}, \leq_P, \bigvee)$$

$+$ internal quasi-topos logic ω-cpo axioms

Theorem

$$\omega\mathbf{Qbs} \simeq \omega\mathbf{Cpo}(\mathbf{Qbs})$$

Characterising $\omega\mathbf{Qbs}$

By local presentability:

$$\omega\mathbf{Cpo} \simeq \mathrm{Mod}(\boldsymbol{\omega}\mathbf{cpo}, \mathbf{Set}) \qquad \mathbf{Qbs} \simeq \mathrm{Mod}(\mathbf{qbs}, \mathbf{Set})$$

essentially algebraic theories

$\boldsymbol{\omega}\mathbf{qbs}$:

$$\boldsymbol{\omega}\mathbf{cpo} \cup \mathbf{qbs} \cup \text{compatibility axiom}$$

Theorem

$$\omega\mathbf{Qbs} \simeq \mathrm{Mod}(\boldsymbol{\omega}\mathbf{qbs}, \mathbf{Set})$$

so $\omega\mathbf{Qbs}$ locally presentable, hence cocomplete

A probabilistic powerdomain

Lebesgue integration:

$$\begin{array}{c}\mathbb{R}\\ \alpha\downarrow\\ X_\bot\end{array} \quad \mapsto \quad \lambda \begin{array}{c}X\\ f\downarrow\\ \mathbb{L}\end{array}. \int_{\alpha^{-1}[X]} f \circ \alpha(x) \lambda(\mathrm{d}x)$$

$\alpha^{-1}[X] \xrightarrow{f\circ\alpha} [0, \infty]$ Borel

Lebesgue measure

$\alpha^{-1}[X]$ Borel

Borel maps and natural order

$$(X_\bot)^{\mathbb{R}} \longrightarrow \mathbb{L}^{\mathbb{L}^X}, \qquad (X_\bot)^{\mathbb{R}} \twoheadrightarrow MX \rightarrowtail \mathbb{L}^{\mathbb{L}^X} \quad (=)$$

continuation monad

where:

$$X \xrightarrow{f} Y, \qquad X \twoheadrightarrow \left(\mathrm{Cl}_\omega f[X], \leq_Y, \mathrm{Cl}_\omega^{Y^{\mathbb{R}}} f \circ [M_X]\right) \rightarrowtail Y \quad (=)$$

A probabilistic powerdomain

$(\mathcal{E}, \mathcal{M}) :=$ (densely strong epi, full mono) factorisation system:

$$X \xrightarrow{\;f\;} Y \quad = \quad X \twoheadrightarrow \left(\mathrm{Cl}_\omega f[X], \leq_Y, \mathrm{Cl}_\omega^{Y^{\mathbb{R}}} f \circ [M_X]\right) \rightarrowtail Y$$

$\mathcal{E}$ closed under:

- products: $e_1, e_2 \in \mathcal{E}_q \implies e_1 \times e_2 \in \mathcal{E}_q$
- lifting: $e \in \mathcal{E} \implies e_\perp \in \mathcal{E}$
- random elements: $e \in \mathcal{E} \implies e^{\mathbb{R}} \in \mathcal{E}$

$\implies$ M strong monad for sampling + conditioning

[Kammar-McDermott'18]

A probabilistic powerdomain

$$\begin{array}{ccc} (X_\perp)^{\mathbb{R}} & \longrightarrow & \mathbb{L}^{\mathbb{L}^X} \\ & \searrow \quad = \quad \nearrow & \\ & MX & \end{array}$$

- M locally continuous
- M commutative
- $M \sum_{n\in\mathbb{N}} X_n \cong \prod_{n\in\mathbb{N}} MX_n$

$\Longrightarrow$ synthetic measure theory model

[Kock'12,
Ścibior et al.'18]

- $MX \cong \left\{ \mu|_{\text{Scott opens}} \,\middle|\, \mu \text{ is s-finite} \right\}$

standard Borel space

Axiomatic domain theory

[Fiore-Plotkin'94, Fiore'96]

Structure

- Total map category: $\omega\mathbf{Qbs}$
- Admissible monos: **Borel-open** map $m : X \rightarrowtail Y$:

strong mono

qbses

$$\forall \beta \in M_Y. \qquad \beta^{-1}[m[X]] \in \mathcal{B}(\mathbb{R})$$

  take Borel-Scott open maps as admissible monos
- $\mathbf{Pos}$-enrichment: pointwise order
- Pointed monad on total maps: the powerdomain

$\Longrightarrow$ model axiomatic domain theory

$\Longrightarrow$ solve recursive domain equations

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