A convenient category for higher-order probability theory

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https://arxiv.org/abs/1701.02547

Thirty-Second Annual ACM/IEEE Symposium on
Logic in Computer Science (LICS)
20–23 June 2017, Reykjavik

What is higher-order probabilistic programming?

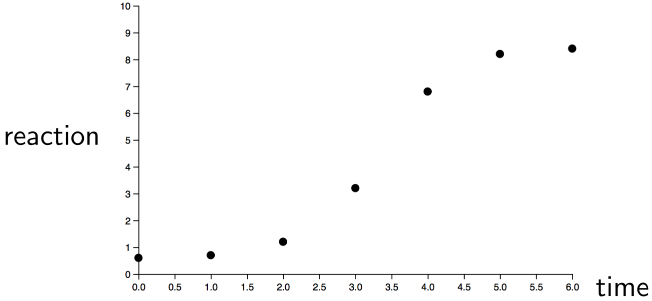
Bayesian data modelling

1. Develop a probabilistic (generative) model.
2. Design an inference algorithm for the model.
3. Using the algorithm, fit the model to the data.

What is higher-order probabilistic programming?

Example

Effect of a drug on a patient, given data:



What is higher-order probabilistic programming?

Generative model

$$
\begin{aligned}
s &\sim \mathsf{normal}(0, 2) \\
b &\sim \mathsf{normal}(0, 6) \\
f(x) &= s \cdot x + b \\
y_i &= \mathsf{normal}(f(i), 0.5) \\
&\qquad \text{for } i = 0 \ldots 6
\end{aligned}
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Conditioning

$y_0 = 0.6, y_1 = 0.7, y_2 = 1.2, y_3 = 3.2, y_4 = 6.8, y_5 = 8.2, y_6 = 8.4$

Predict f?

What is higher-order probabilistic programming?

Bayesian inference

$$P(s, b|y_0, \ldots, y_6) = \frac{P(y_0, \ldots, y_6|s, b) \cdot P(s, b)}{P(y_0, \ldots, y_6)}$$

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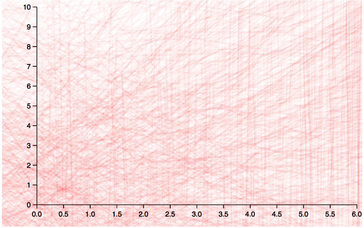

Prior

What is higher-order probabilistic programming?

Bayesian inference

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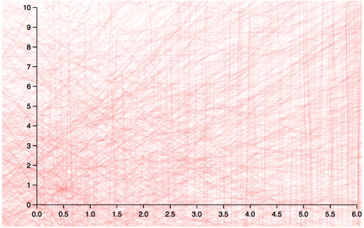
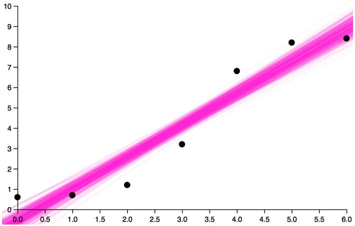


Prior



Posterior

What is higher-order probabilistic programming?

Probabilistic programming models

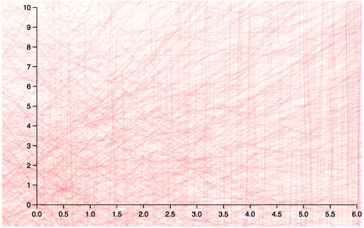
1. Develop a probabilistic (generative) model.
   Write a program.
2. ~~Design an inference algorithm for the model.~~
3. Using the built-in algorithm, fit the model to the data.

What is higher-order probabilistic programming?

In Anglican [Wood et al.'14]

```
(let [s (sample (normal 0.0 2.0))
      b (sample (normal 0.0 6.0))
      f (fn [x] (+ (* s x) b))]
```

```
(predict :f f))
```



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  (observe (normal (f 1.0) 0.5) 2.5)
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  (predict :f f))
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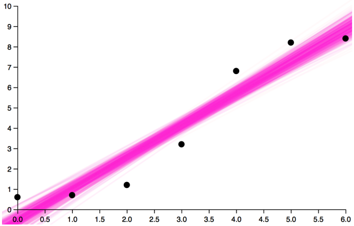

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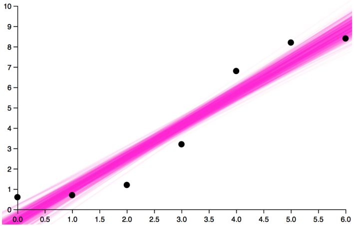

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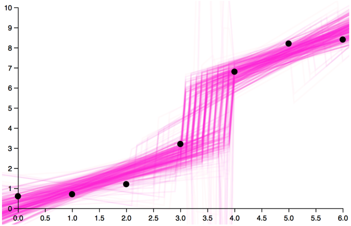

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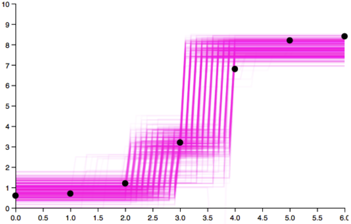

Semantics for probabilistic languages

Specification

- Continuous distributions.

Semantics

Category of measurable spaces

Semantics for probabilistic languages

Specification

- Continuous distributions.
- Distributions over higher-order data.
- Higher-order "glue".

Semantic challenge

Category of measurable spaces is **not** Cartesian closed!

Semantics for probabilistic languages

Specification

- Continuous distributions.
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Semantic challenge

Category of measurable spaces is **not** Cartesian closed!

Theorem (Aumann'61)

The set $\mathbf{Meas}(\mathbb{R}, \mathbb{R})$ *cannot be made into a measurable space with*

$$eval : \mathbf{Meas}(\mathbb{R}, \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$$

measurable.

Contribution

Quasi-Borel spaces

A Cartesian closed category for higher-order probability theory:

- Sets-with-structure ('random elements') and structure preserving functions.
- Well-behaved and concrete categorical structure.
- Well-behaved probabilistic structure: randomization and de Finetti's Theorem

Brief measure theory

Measures subsets of $\mathbb{R}$

Borel subsets $\mathcal{B}(\mathbb{R})$ as closure under:

- Intervals $[a, b]$.
- Countable unions.
- Complements.

$\varphi : \mathbb{R} \to \mathbb{R}$ is **measurable** when:

$$B \in \mathcal{B}(\mathbb{R}) \qquad \Longrightarrow \qquad \varphi^{-1}[B] \in \mathcal{B}(\mathbb{R})$$

Measure theory generalises to σ-algebras over arbitrary sets.

Source of randomness

Key idea

Propagating randomness from discrete and continuous sampling:

$$\alpha : \mathbb{R} \to X$$

along "random elements":

- for measurable spaces: **derived** through measurable functions;
- for quasi-Borel spaces: **axiomised** through structure.

The category $\mathbf{Qbs}$

Objects

A **quasi-Borel space** $X = \left\langle |X|, X^{\mathbb{R}} \right\rangle$ consists of:

- a **carrier set** X;
- a set of **random elements** $X^{\mathbb{R}} \subseteq |X|^{\mathbb{R}}$

such that the random elements are closed under:

The category **Qbs**

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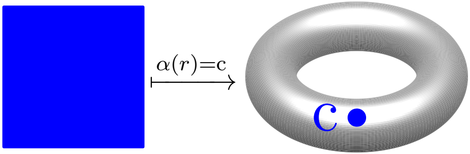

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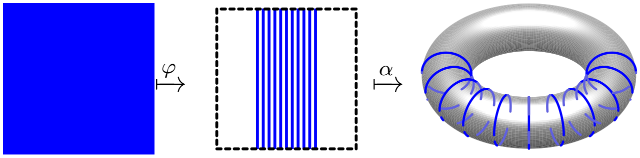

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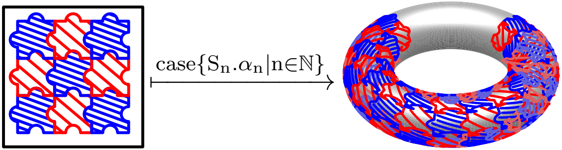
- countable measurable case split.



The category $\mathbf{Qbs}$

Morphisms $f : X \to Y$

Functions $f : |X| \to |Y|$ such that:

$$\alpha \in X^{\mathbb{R}} \qquad \Longrightarrow \qquad f \circ \alpha \in Y^{\mathbb{R}}$$

Categorical structure

Free and cofree spaces

Equip a set $A \in \mathbf{Set}$ with:

$$(\mathsf{Free}\quad A)^{\mathbb{R}} := \left\{\mathsf{case}\,\{S_n.\underline{a}_n | n \in \mathbb{N}\} \middle| (S_n) \text{ a measurable partition}\right\}$$
$$(\mathsf{Cofree}A)^{\mathbb{R}} := A^{\mathbb{R}}$$

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "\mathsf{Free}"] \arrow[r, bend right, "\mathsf{Cofree}"'] & \mathbf{Qbs} \arrow[l] \end{tikzcd}$$

Categorical structure

Measurable spaces

Adjunction with measurable spaces ($M \in \mathbf{Meas}$, $X \in \mathbf{Qbs}$):

$$(\mathbf{Qbs}M)^{\mathbb{R}} := \mathbf{Meas}(\mathbb{R}, M)$$
$$\Sigma_{(\mathbf{Meas}X)} := \left\{ B \subseteq X \middle| \forall \alpha \in X^{\mathbb{R}}, \alpha^{-1}[X] \in \mathcal{B}(\mathbb{R}) \right\}$$

$$\begin{tikzcd}
\mathbf{Qbs} \arrow[rr, bend left, "\mathbf{Meas}"] \arrow[rr, phantom, "\perp"] \arrow[dr] & \arrow[d, phantom, "="] & \mathbf{Meas} \arrow[ll, bend left, "\mathbf{Qbs}"] \arrow[dl] \\
& \mathbf{Set} &
\end{tikzcd}$$

NB: $\mathbf{Meas} \circ \mathbf{Qbs}X = X$ for standard Borel spaces X.

Categorical structure

Products

Correlated random elements:

$$(X \times Y)^{\mathbb{R}} := \left\{ r \mapsto \left\langle \alpha(r), \beta(r) \right\rangle \middle| \alpha \in X^{\mathbb{R}}, \beta \in Y^{\mathbb{R}} \right\}$$

Categorical structure

Function spaces

$$\left|Y^X\right| := \mathbf{Qbs}(X, Y)$$
$$(Y^X)^{\mathbb{R}} := \left\{ f : \mathbb{R} \to \left|Y^X\right| \,\middle|\, \text{uncurry}\, f \in \mathbf{Qbs}(\mathbb{R} \times X, Y) \right\}$$

NB: The exponential $X^{\mathbb{R}}$ is the space of random elements.

Categorical structure

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More structure

Coproducts, limits, colimits, ...

Probabilistic structure

Probability distributions on X

Pairs $\langle \alpha, \mu \rangle$:

- random element $\alpha \in X^{\mathbb{R}}$
- probability distribution μ on $\mathbb{R}$

Too intensional, e.g.:

$$(x \mapsto \min(0, \max(1, \quad x)), \mathbf{U_{[0,1]}})$$

vs.

$$(x \mapsto \min(0, \max(1, 1-x)), \mathbf{U_{[0,1]}})$$

Probabilistic structure

Quotienting distributions

$\langle \alpha, \mu \rangle \sim \langle \beta, \nu \rangle$ when $\alpha_* \mu = \beta_* \nu$ in $\mathbf{Meas} X$ and set

- $|PX|$: $\sim$-equivalence classes of probability distributions.
- $(PX)^{\mathbb{R}} :=$
  $\left\{ r \mapsto [\alpha, k(r)] \middle| \; k : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0,1] \text{ is a Markov kernel } \right\}$

NB: PX coincides with Giry on standard Borel spaces X

Contribution

Quasi-Borel spaces

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Thank you!

FAQ

Difference with previous category [LICS'16]

$$\begin{tikzcd}
\mathbf{SMeas} \arrow[rr, hook, "\mathbf{Qbs}"] \arrow[dr, hook, "\mathbf{y}X=\mathbf{SMeas}(-,X)"'] & {=} & \mathbf{Qbs} \arrow[dl, hook', "\mathbf{Meas}(-,\mathbf{Meas}X)"] \\
& {[\mathbf{SMeas}^{\mathrm{op}}, \mathbf{Set}]_{\prod_\omega}} &
\end{tikzcd}$$

Every $Q \in [\mathbf{SMeas}^{\mathrm{op}}, \mathbf{Set}]_{\prod_\omega}$ has the data:

- $Q\mathbb{1}$: elements.
- $Q\mathbb{R}$: random elements.
- $eval : Q\mathbb{R} \xrightarrow{(Q\underline{r})_{r\in\mathbb{R}}} \prod_{r\in\mathbb{R}} Q\mathbb{1}$.

A quasi-Borel space is fully determined by this data.

Thank you!

FAQ

Why call them "quasi-Borel spaces"?

$$\begin{tikzcd}
\mathbf{SMeas} \arrow[rr, hook, "\mathbf{Qbs}"] \arrow[dr, hook, "\mathbf{y}X = \mathbf{SMeas}(-,X)"'] & \arrow[d, phantom, "="] & \mathbf{Qbs} \arrow[dl, hook', "\mathbf{Meas}(-,\mathbf{Meas}X)"] \\
& {[\mathbf{SMeas}^{\mathrm{op}}, \mathbf{Set}]_{\prod_\omega}} &
\end{tikzcd}$$

Analogous to Spanier's quasi-topological spaces that arise similarly:

- Sheaf (in this case): countable product preservation
- Separatedness: $eval : Q\mathbb{R} \xrightarrow{(Q\underline{r})_{r\in\mathbb{R}}} \prod_{r\in\mathbb{R}} Q\mathbb{1}$ injective

Thank you!

FAQ

What about recursion?

- Sub-probability distributions can deal with first-order iteration
- We are currently working on type recursion.