A convenient fibration for dependently-typed probability theory

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slides:

paper:

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What have we done?

Semantic **model** for **dependent types** $\Gamma, x : A \vdash B$

best case scenario: (split full comprehension category [Jacobs'93])

$\Gamma \vdash (x : A) \to B$
dependent functions
(Π-types)

$\Gamma \vdash (x : A) \times B$
dependent pairs
(Σ-types)

$\Gamma \vdash M =::= K$
propositional equality
(extensional)

$\Gamma \vdash \{x : A | \varphi\}$
subspaces
(refinement types)

$a, b : \mathbb{R} \vdash (a, b]$
continuous spaces

where **construction** means (quasi-)**measurable map**

What have we done?

Fibred Giry monad (split, in fact) [Lawvere'62, Giry'80]

$$\Gamma \vdash \mathbf{D}A \qquad \Gamma \vdash \mathbf{P}A \coloneqq \{\mu : \mathbf{D}A \mid \mu(A) = 1\} \qquad (\Gamma \vdash A)$$

distributions / measures (non-normalised) — probability distributions

supporting continuous probability with the usual properties:
(fibred variant of Kock's ['12] **synthetic measure theory**)

$$a, b : \mathbb{R}, a < b \vdash \mathbf{U}_{(a,b]} : \mathbf{P}(a, b]$$

uniform distribution

$$\Gamma, \mu : \mathbf{D}A, f : A \to [0, \infty], \nu : \mathbf{D}B, g : B \to [0, \infty] \vdash$$

$$\int \mathrm{d}\mu f \int \mathrm{d}\nu g = \iint (\mu \otimes \nu)(\mathrm{d}x, \mathrm{d}y)(f\,x) \cdot (f\,y) = \int \mathrm{d}\nu g \int \mathrm{d}\mu f$$

(fibred Fubini's theorem)

What have we done?

Dependently-typed probability

Case study: probabilistic concept inhabiting inherently dependent space:

$$\Gamma \vdash \mathbb{E} : \quad (\mu : \mathbf{D}\Omega) \to \overbrace{(H : \Omega \to \Theta)}^{\textbf{observation}} \to \overbrace{(f : \mathcal{L}^1(\Omega, \mu))}^{\mu\textbf{-integrable r.v.}} \to \underbrace{\left\{ g : \mathcal{L}^1(\Omega, \mu_H) \middle| g = \mathbb{E}_\mu [f \mid H] \right\}}_{\textbf{conditional expectation}} \qquad \text{(standard } \Theta\text{)}$$

by extracting **construction** from standard proof

What have we done?

Best possible situation:

- Semantic model for dependent types (Π, Σ, extensional identity, subspaces)
  (split fibration)
- Semantic model for dependently-typed probability
  (split distribution and probability monads)

Case study

- dependently-typed probability
  (measurability of conditional expectation under regularity assumptions)

Why do it?

Dependent types trim ill-defined edge cases

- **Urn models** indexed by their total number of balls: [Jacobs'22]

$$n : \mathbb{N} \vdash \mathrm{Urn}_n \coloneqq \{(b, r) : \mathbb{N} \times \mathbb{N} | b + r = n\}$$

- Avoid '**almost-sure** safety', obtain actual '**safety**':

$$a, b : \mathbb{R} \vdash \left(w \sim \mathbf{U}_{(0,1]}; \frac{1}{w}\right) : \mathbf{P}[1, \infty)$$

Why do it?

Community interest

- Stream-sampling for probabilistic programming using continuous functions
  via **compactly generated Hausdorf spaces** [Dahlqvist, Silva, and Smith'23]
- **Synthetic probability** for dependent types with Markov fibrations [Rischel'26]
- Probabilistic **graphical models** with the codomain fibration
  [St Clere Smithe '24, '25]

Personal interest

- Dependently-typed **Probabilistic programming** and **Bayesian inference** (WIP)
  (use Brady's ['21] Idris 2)
- Unsatisfactory early version of use-case using **HOL** and **refinement types**
  CIRM tutorial [Kammar'22, unpublished]

Why is it tricky?

Aumann's theorem ['61]

No simply-typed measurable-function measurable space, let alone dependent-functions

Locally Cartesian-closed categories

Codomain fibrations of sheaf models for probability model full dependent types

several models for classical continuous probability

quasi-Borel spaces and their precursor [Staton et al. '16+'17]

probability sheaves [Simpson'17+'24]

nominal techniques [Li, Aytac, Johnson-Freyd, Ahmed, Holtzen'24]

still need a fibred probability monad!

(codomain fibration also not **split**—

inconvenient for external development in Staton et al. and Simpson models)

Why is it tricky?

What should $\Gamma \vdash \mathbf{P}A$ be?

Non-answer:

$$\gamma : \Gamma \vdash \text{✗}A := \left\{\mu : \mathbf{P}(\Gamma, x : A) \middle| \mu(\mathrm{d}\gamma', x \mapsto a)\text{-a.s. } \gamma' = \gamma\right\}$$

✗A is a monad on each fibre, but reindexing/substitution does not preserve it:

$$(\text{✗}A)\,[\theta] \ncong \text{✗}(A\,[\theta])$$

✗A isn't **fibred monad**

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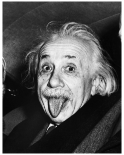
Danel and I listened to **Einstein**:

> "If you are out to describe the truth, leave elegance to the tailor."



We started developing semantics for dependent types with non-fibred monads.

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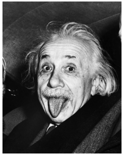
"If you are out to describe the truth, leave elegance to the tailor."

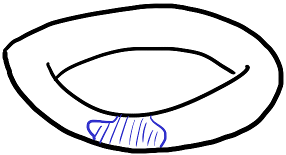
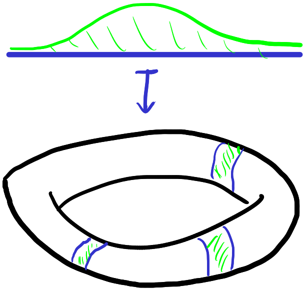
"Danel and Ohad, your pain threshold is much too high for your own good."



Talk structure

- Motivation
- quasi-Borel spaces & families
- fibred probability monads
- dependently-typed probability: conditional expectation

Quasi-Borel spaces recap: core idea [Staton et al.'17]

	measurable spaces	**quasi Borel** spaces
		
primitive notions:	points & **admissible events** (measurable subsets)	points & **admissible random elements**
derived notions:	**admissible random elements** $\alpha : \mathbb{R} \to M$	**admissible events** $E \in \mathcal{B}_X$

Metaphorology[1] $\mathcal{R}$

over set $\underline{X}$ of **points**: subset of functions—**admissible random elements**—
$\alpha : \mathbb{R} \to \underline{X}$ closed under:

constants:

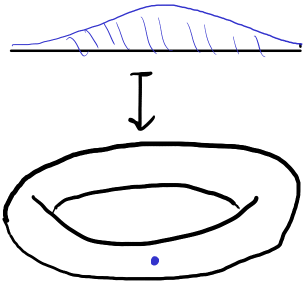
$$\frac{x \in \underline{X}}{\underline{x} := (\lambda r.x) \in \mathcal{R}}$$

precomposition:

$$\frac{\alpha \in \mathcal{R} \qquad \varphi \in \mathbf{Meas}(\mathbb{R}, \mathbb{R})}{(\alpha \circ \varphi) \in \mathcal{R}}$$

recombination:

$$\frac{I \subseteq \mathbb{N} \quad E : I \to \mathcal{B}_{\mathbb{R}} \quad \mathbb{R} = \biguplus_{i \in I} E_i \quad \alpha : I \to \mathcal{R}}{[E_i.\alpha_i]_{i \in I} := (\lambda r \in E_i.\alpha_i r) \in \mathcal{R}}$$



[1] $\mu\varepsilon\tau\alpha$ ('meta', across) and $\varphi\varepsilon\rho\omega$ ('phero', to carry).

Quasi-Borel spaces: core definitions [Staton et al.'17]

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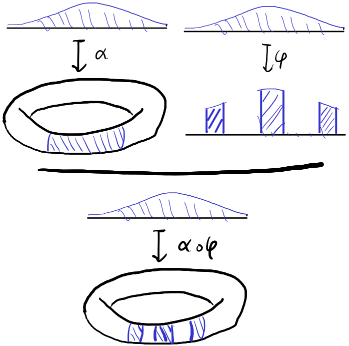

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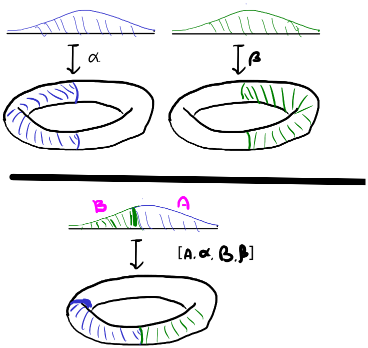

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Quasi Borel space (qbs) X

set of points $\underline{X}$ equipped with
a metaphorology $\mathcal{R}_X$ over it

[1] $\mu\varepsilon\tau\alpha$ ('meta', across) and $\varphi\varepsilon\rho\omega$ ('phero', to carry).

Quasi-Borel families: core definitions [this work]

Γ-fibred metaphorology $\mathcal{R}^{-}$ over $\gamma : \underline{\Gamma} \vdash \underline{X}_\gamma$

$\mathcal{R}_\Gamma$-indexed family of subsets $\boldsymbol{\upsilon} : \mathcal{R}_\Gamma \vdash \mathcal{R}^{\boldsymbol{\upsilon}} \subseteq (r : \mathbb{R}) \Rightarrow \underline{X}_{\boldsymbol{\upsilon}\, r}$ closed under:

fibred constants:

$$\frac{\gamma \in \underline{\Gamma} \qquad a \in \underline{X}_\gamma}{\underline{a} \coloneqq (\lambda r.a) \in \mathcal{R}^{\underline{\gamma}}}$$

fibred precomposition:

$$\frac{\alpha \in \mathcal{R}^{\boldsymbol{\upsilon}} \qquad \varphi \in \mathbf{Qbs}(\mathbb{R}, \mathbb{R})}{(\alpha \circ \varphi) \in \mathcal{R}^{\boldsymbol{\upsilon} \circ \varphi}}$$

fibred recombination:

$$\frac{I \subseteq \mathbb{N} \qquad E : I \to \mathcal{B}_{\mathbb{R}} \qquad \mathbb{R} = \biguplus_{i \in I} E_i \qquad \alpha : (i : I) \to \mathcal{R}^{\boldsymbol{\upsilon}_i}}{[E_i.\alpha_i]_{i \in I} \in \mathcal{R}^{[E_i.\boldsymbol{\upsilon}_i]_{i \in I}}}$$

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Quasi Borel family (qbf) $\Gamma \vdash X$

a family $\gamma : \underline{\Gamma} \vdash \underline{X}_{\gamma}$ equipped with a Γ-fibred metaphorology $\mathcal{R}_X$ over $\underline{X}$.

Examples

- $$\frac{\Gamma, X \text{ qbses}}{\Gamma \vdash X \text{ qbf}} \qquad \gamma : \underline{\Gamma} \vdash \underline{X}_\gamma := X \qquad \boldsymbol{v} : \mathcal{R}_\Gamma \vdash \mathcal{R}^{\boldsymbol{v}}_X := \mathcal{R}_X$$
- Intervals are qbfs:

$$a, b : [-\infty, \infty] \vdash [a, b], [a, b), (a, b], (a, b)$$

with $\mathcal{R}^{a \mapsto \alpha, b \mapsto \beta}$ those measurable functions $\varphi \in \mathbf{Qbs}(\mathbb{R}, [0, \infty])$ such that, for all $r \in \mathbb{R}$, $\varphi\, r$ is in the interval delimited by $\alpha\, r$ and $\beta\, r$.

- For measurable map $\begin{matrix} X \\ d\downarrow \\ \Gamma \end{matrix}$, the **preimage** qbf:

$$\gamma : \Gamma \vdash d^{-1}[\gamma] \qquad \mathcal{R}^{\boldsymbol{v}}_{d^{-1}[-]} := \begin{pmatrix} \mathcal{R}_X \\ d\circ\downarrow \\ \mathcal{R}_\Gamma \end{pmatrix}^{-1} [\boldsymbol{v}] \subseteq \mathcal{R}_X$$

Quasi-measurable maps

Qbf map $(\theta \vdash f) : (\gamma : \Gamma \vdash X_\gamma) \to (\delta : \Delta \vdash Y_\delta)$

a family map $(\theta \vdash f) : (\gamma : \underline{\Gamma} \vdash \underline{X}_\gamma) \to (\delta : \underline{\Delta} \vdash \underline{Y}_\delta)$, i.e.:

$$\theta : \underline{\Gamma} \to \underline{\Delta} \qquad \gamma : \Gamma \vdash f_\gamma : \underline{X}_\gamma \to Y_{\theta\gamma}$$

such that:

- $\theta : \Gamma \to \Delta$ in $\mathbf{Qbs}$, i.e., preserves random elements:

$$\boldsymbol{v} \in \mathcal{R}_\Gamma \implies (\theta \circ \boldsymbol{v}) \in \mathcal{R}_\Delta$$

- f preserves fibred random elements:

$$\mathcal{R}_X^{\boldsymbol{v}} \ni \alpha \qquad \implies \qquad (f \overset{\theta}{\circ} \alpha) \coloneqq (\lambda r. f_{\theta\, r}(\alpha\, r)) \in \mathcal{R}_Y^{\theta \circ \boldsymbol{v}}$$

Type-theory model

Theorem

The category of qbfs and their maps, $\mathbf{Qbf}$, equipped with the functor:

$$\mathsf{Index} : (\Gamma \vdash X_{\_}) \mapsto \Gamma : \mathbf{Qbf} \to \mathbf{Qbs}$$

is a split fibred category.
It supports:

- *split Π-types;*
- *Σ-types;*
- *split extensional propositional equality; and*
- *split subspaces.*

The **Grothendieck construction** *and the* **preimage** *form a fibred adjoint equivalence between* Index *and the codomain fibration $\mathbf{Qbs}^{\to} \to \mathbf{Qbs}$.*

Talk structure

- Motivation
- quasi-Borel spaces & families
- fibred probability monads
- dependently-typed probability: conditional expectation

Lebesgue integration of random elements

measurable map:

$$\int \mathrm{d}[-] \coloneqq \left(\lambda f.\lambda g.\int_{\mathrm{Dom}(f)} \mathrm{d}x(g \circ f)\right) : \underbrace{(\mathbb{R} \to X^{\perp})}_{=:\mathbf{R}X} \to \underbrace{(X \to \mathbb{W}) \to \mathbb{W}}_{=:\mathbf{K}X}$$

NB:

- measurable **partial functions** $f : X \rightharpoonup Y$ are measurable functions $f : X \rightharpoonup (Y \amalg \{\perp\}) =: Y^{\perp}$
- space of **weights**: $\mathbb{W} \coloneqq [0, \infty]$

Simply-typed distributions: recap [cf. Vákár et al.'19]

Lebesgue integration of random elements

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Distribution monad $\mathbf{D}X$

Strong epi-mono factorisation of $\int \mathrm{d}[-]$:

$$\int \mathrm{d}[-] : \mathbf{R}X \overset{[-]}{\twoheadrightarrow} \mathbf{D}X \overset{\int \mathrm{d}}{\rightarrowtail} \mathbf{K}X$$

concretely:

$$\underline{\mathbf{D}X} \coloneqq \{[\alpha] \mid \alpha : \mathbb{R} \rightharpoonup X\}$$

$$\mathcal{R}_{\mathbf{D}X} \coloneqq \left\{\alpha : \mathbb{R} \to \mathbf{D}X \;\middle|\; \begin{array}{l} \exists \beta \in \mathcal{R}_{\mathbf{R}X}. \\ \alpha = \lambda r.[\beta\, r] \end{array}\right\}$$

Lebesgue integration of fibred random elements

measurable map (monomorphism in $\mathbf{Qbs}$):

$$\Gamma \vdash \int \mathrm{d}[-] : \underbrace{(\mathbb{R} \to X^{\perp})}_{=:\mathbf{R}X} \to \underbrace{(X \to \mathbb{W}) \to \mathbb{W}}_{=:\mathbf{K}X}$$

Fibred distribution monad $\mathbf{D}X$

Fibred strong epi-mono factorisation of $\int \mathrm{d}[-]$:

$$\Gamma \vdash \int \mathrm{d}[-] : \mathbf{R}X \xrightarrow{[-]} \mathbf{D}X \overset{\int \mathrm{d}}{\rightarrowtail} \mathbf{K}X$$

concretely:

$$\gamma : \underline{\Gamma} \vdash \underline{\mathbf{D}X}_{\gamma} := \{[\alpha] \mid \alpha : \mathbb{R} \rightharpoonup X_{\gamma}\} \qquad \boldsymbol{\upsilon} : \mathcal{R}_{\Gamma} \vdash \mathcal{R}^{\boldsymbol{\upsilon}}_{\mathbf{D}X} := \left\{ \alpha : \mathbb{R} \to \mathbf{D}X \;\middle|\; \begin{array}{l} \exists \beta \in \mathcal{R}^{\boldsymbol{\upsilon}}_{\mathbf{R}X}. \\ \alpha = \lambda r.[\beta\, r] \end{array} \right\}$$

Synthetic measure theory

Theorem

$\mathbf{D}$ and $\mathbf{P}X \coloneqq \left\{\mu : \mathbf{D}X \,\middle|\, 1 = \mu X \coloneqq \int \mathrm{d}\mu 1\right\}$ are commutative split monads on Index*.*
The fibres of $\mathbf{D}$ satisfy Kock's ['12] synthetic measure theory axioms—there are canonical isomorphisms:

$$\Gamma \vdash \mathbf{D}\left(\coprod_{i\in I} A^i\right) \xrightarrow{\cong} \bigtimes_{i\in I} \mathbf{D}(A^i) \qquad (I \text{ countable})$$

Consequences

Derived notions of, e.g.:

semi-ring of scalars: $\mathbb{W} = [0, \infty]$; expectation; disintegration; almost-sure equality; density / absolute continuity / Radon-Nikodym derivatives; etc. [Kock'12]

compatible with measure-theoretic concepts [Ścibior et al. 18]

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Use case

Study spaces of integrable random variables—**Lebesgue** space quasi-Borel **family**:

$$\frac{\Gamma \vdash \mu : \mathbf{P}\Omega \qquad \Gamma \vdash p : [1, \infty)}{\Gamma \vdash \mathcal{L}^p(\Omega, \mu) \coloneqq \left\{ f : \Omega \to \mathbb{R} \,\middle|\, f \in \mathrm{Dom}\left(\int \mathrm{d}\mu\right) \wedge \int \mathrm{d}\mu \, |f| < \infty \right\}}$$

Different from the quotiented **Banach** spaces:

$$\mathrm{L}^{p_\gamma}(\Omega_\gamma, \mu_\gamma) \coloneqq \mathcal{L}^{p_\gamma}(\Omega_\gamma, \mu_\gamma)/(\mu_\gamma\text{-a.s. equality})$$

Conditional expectation

Central higher-order concept to modern probability:

$$\frac{\Gamma \vdash \mu : \mathbf{D}\Omega \qquad \Gamma \vdash \overset{\text{observation}}{H : \Omega \to \Theta} \qquad \Gamma \vdash \overset{\text{latent rv}}{f : \mathcal{L}^1(\Omega, \mu)} \qquad \Gamma \vdash \overset{\text{observed rv}}{g : \mathcal{L}^1(\Theta, \mu_H)}}{\Gamma \vdash (g = \mathbb{E}_\mu [f|H]) := \qquad \forall w : \Theta \to \mathbb{W}.}$$

$$\int \mu_H(\mathrm{d}\theta) g(\theta) \cdot w(\theta) = \int \mu(\mathrm{d}\omega) f(\omega) \cdot w(H\,\omega) : \mathbf{Prop}$$

Theorem

Let $\Gamma \vdash \Omega$ be any qbf and $\vdash \Theta$ a standard Borel space. We have a measurable dependent function:

$$\Gamma \vdash \mathbb{E} : (\mu : \mathbf{P}\Omega) \to (H : \Omega \to \Theta) \to (f : \mathcal{L}^1(\Omega, \mu)) \to \left\{ g : \mathcal{L}^1(\Theta, \mu_H) \middle| g = \mathbb{E}_\mu [f|H] \right\}$$

Proof consists of noticing the 'constructive' content in the standard proof [Williams'91]

What have we done?

Best possible situation:

- Semantic model for dependent types (Π, Σ, extensional identity, subspaces)
  (split fibration)
- Semantic model for dependently-typed probability
  (split distribution and probability monads)

Case study

- dependently-typed probability
  (measurability of conditional expectation under regularity assumptions)

Future work

Dependently-typed accounts of:

- disintegration;
- martingale theory;
- stochastic differential equations
  (NB: generalized conditional expectation may need further assumptions)
- universe constructions
  (already have $\mathbf{Sbs} := \mathcal{B}_{\mathbb{R}}$; $\mathbf{Pol}$ and $\mathbf{CSepMet}$ via Effros space)
- implications to Bayesian inference implementations;
- implications to probabilistic programming semantics
  (operational and denotational)

Thank you!