Symmetric Programming and Reasoning

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Laboratory for Foundations of Computer Science


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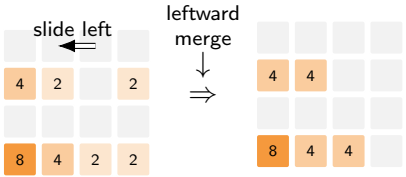
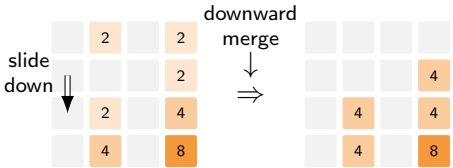
Example: 2048 game[1] [Cirulli'14]





[1] Available to play: `https://play2048.co`

2048 in Idris 2 [Brady'21]

Merging left goes with the grain

```
Row, Board : Nat -> Type
Row    n = Vect n Int
Board n = Vect n (Row n)

(.pad0) : Row n -> Row (1 + n)
xs.pad0 = replace {p = Row}
                  (plusCommutative _ _)
          $ xs ++ [0]

mergeRow : Row n -> Row n
mergeRow []                = []
mergeRow (     0 :: xs)    = (mergeRow xs).pad0
mergeRow (     x :: [])    = [x]
mergeRow (x :: 0 :: xs)    = (mergeRow (x :: xs)).pad0
mergeRow (x :: y :: xs) with (x == y)
                        _ | True  = 2*x :: (mergeRow xs).pad0
                        _ | False =   x :: mergeRow (y :: xs)
mergeLeft : Board n -> Board n
mergeLeft = map mergeRow
```

Merging in 2048

Going against the grain

merging **up/down** incompatible with **inductive structure**

⇓

use a different representation

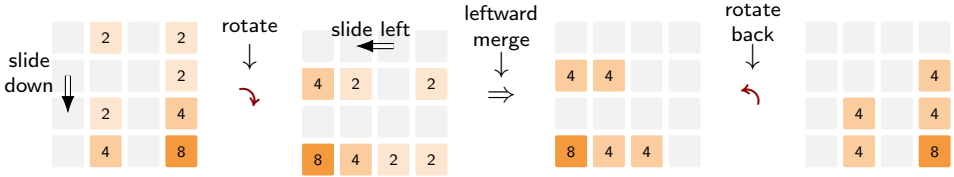
⇓

parameters galore![2]

```
  var vector     = this.getVector(direction);
  var traversals = this.buildTraversals(vector);
// ...
GameManager.prototype.getVector =
 function (direction) {// Vectors representing tile movement
  var map = { 0: { x:  0, y: -1 },  // Up
              1: { x:  1, y:  0 },  // Right
              2: { x:  0, y:  1 },  // Down
              3: { x: -1, y:  0 }}; // Left
  return map[direction];
};
```

[2] See the function `GameManager.prototype.move`:
https://github.com/gabrielecirulli/2048/blob/478b6ec346e3787f589e4af751378d06ded4cbbc/js/game_manager.js#L129.

Easy 2048 implementation using symmetry

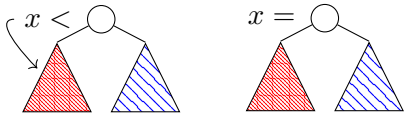


Anecdotal evidence with half-a-dozen UG students

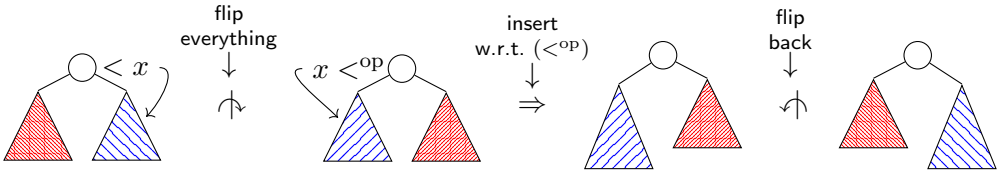
- ✗ buggy parametric solution
- ✓ mostly right symmetric solution

Example: binary search tree insertion using **symmetry**

Interesting cases



Symmetric case



NB: Yes, silly performance. We'll improve this later.

Symmetric reasoning without-loss-of-generality

Theorem (small pigeonhole principle[3])

If three balls are painted red or blue, at least two have the same color.

Proof

Without loss-of-generality, ball 1 is blue. If balls 2 and 3 are the same colour, pick them and conclude. If they have different colours, **without loss-of-generality**, ball 2 is blue. Pick balls 1 and 2 and conclude. ■

Valid proof, but dubious nesting of wlog.

Dijkstra [EWD1223, '95]: typical wlog proofs omit relevant details.
Presents a wlog principle for predicates over pairs.

Harrison ['09]: there ought to be a wlog principle.
Presents 6 wlog HOL light tactics in different domains (goal: formalisation fidelity)

[3] Adapted from the Wikipedia entry *wlog*.

Symmetric reasoning without-loss-of-generality

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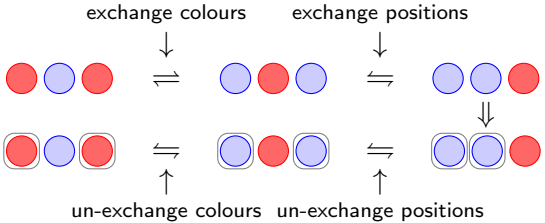
today: symmetric wlog as group-action **conjugation** of **core constructions**



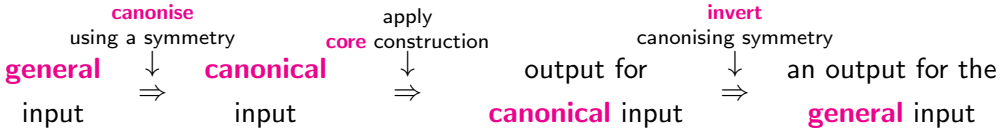
[3] Adapted from the Wikipedia entry *wlog*.

Symmetric programming

Goal: exploit **symmetry** for **computation** and **reasoning**

w.l.o.g. **anatomy**

w.l.o.g. = (input & output **symmetries**)+(**canonising** symmetries)+(**core** construction)



symmetry = **group** + **action**

Contribution

Symmetric programming

Exploit symmetries for computation and reasoning—**avoid symmetric cases**.
Mathematical foundations for symmetric programming semantics:

- **Abstractions** for symmetric programming
- **Simply-typed** reasoning about equivariance
- **Dependently-typed** and mathematical reasoning
- User-defined **data-types** through initial algebra semantics and induction principles
  - Simply-typed data-types
  - Dependently-typed/indexed data-types

Talk structure

- Introduction
- Simply-typed symmetric programming
- Further contributions
- Conclusion

Group theory basics

I'm **not** going to do:

Group G (textbook definition)

monoid $G = \langle \underline{G}, (*), e \rangle$ with inverses:

$$g * g^{-1} = e = g^{-1} * g$$

(Non)-intuition

Only helps you if you know about monoids, but anyway obscures intuition.

Basic group theory

Group action $\langle G, A \rangle$

Group $G = \langle \underline{G}, \ldots \rangle$

G-**action** $A = \langle \underline{A}, \ldots \rangle$

Intuition

- **symmetries**: $\underline{G}$ elements
- x **symmetric to** y **via** g

action and group axioms mean:

'**being symmetric**' is a
proof-relevant equivalence relation tracked by
symmetries

Rotations $\langle \overset{G:=}{\mathbf{C}_4}, \overset{=:A}{\mathsf{Dir}} \rangle$

$\underline{\mathbf{C}_4} := \{\curvearrowleft, \curvearrowright, \curvearrowright, \circlearrowright\}$

acting on

$\underline{\mathsf{Dir}} := \{\Leftarrow, \Rightarrow, \Uparrow, \Downarrow\}$

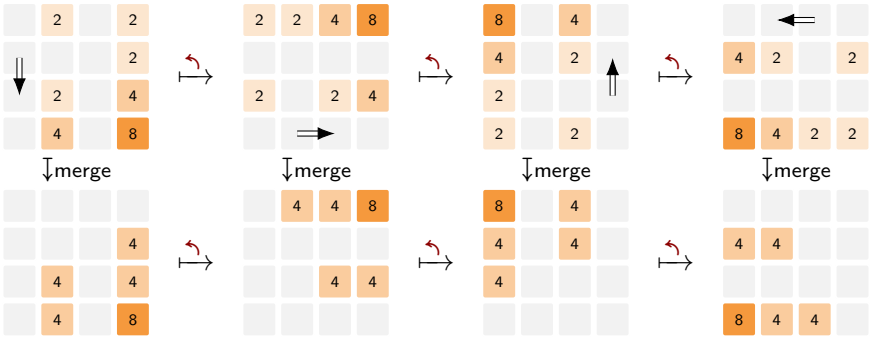
$\Uparrow$ symmetric to $\Leftarrow$ via $\curvearrowleft$
$\Downarrow$ symmetric to $\Leftarrow$ via $\curvearrowright$
$\Rightarrow$ symmetric to $\Leftarrow$ via $\curvearrowright$
$\Leftarrow$ symmetric to $\Leftarrow$ via $\circlearrowright$

Group theory basics

Equivariant map $\varphi : A \circledast\!\!\longmapsto B$ between G-actions A, B

$$\frac{x, \quad y \text{ symmetric via } g}{f\,x, f\,y \text{ symmetric via } g}$$

Example: merge : Dir $\times$ Board$_n$ $\circledast\!\!\longmapsto$ Board$_n$



Basic group theory

Group action $\langle G, A \rangle$

Group $G = \left\langle \underline{G}, (*), e, (-)^{-1} \right\rangle$:

$\underline{G}$(**carrier** set) $\quad \dfrac{g, h : \underline{G}}{g * h : \underline{G}}$(**multiplication**/**composition**)

$\dfrac{}{e : \underline{G}}$(**unit**/**identity**/**neutral**) $\quad \dfrac{g : \underline{G}}{g^{-1} : \underline{G}}$(**inverse**)

such that:

$$\mathsf{g}, \mathsf{h}, \mathsf{k} : \underline{G} \vdash (\mathsf{g} * \mathsf{h}) * \mathsf{k} = \mathsf{g} * (\mathsf{h} * \mathsf{k}) \qquad \text{(associativity)}$$

$$\mathsf{g} : \underline{G} \vdash \mathsf{g} * e = \mathsf{g} = e * \mathsf{g} \qquad \text{(neutrality)}$$

$$\mathsf{g} : \underline{G} \vdash \mathsf{g} * \mathsf{g}^{-1} = e = \mathsf{g}^{-1} * \mathsf{g} \qquad \text{(inverses)}$$

G-**action** $A = \langle \underline{A}, \ldots \rangle \ldots$

Rotations $\langle \overset{G:=}{\mathbf{C}_4}, \overset{=:A}{\mathsf{Dir}} \rangle$

$\underline{\mathbf{C}_4} := \{\circlearrowleft, \circlearrowright, \circlearrowright, \circlearrowright\}$

$\circlearrowleft * \circlearrowleft := \circlearrowright \quad \circlearrowleft * \circlearrowright := \circlearrowright$

$\circlearrowleft * \circlearrowright := \circlearrowright \quad \circlearrowleft * \circlearrowright := \circlearrowleft$

group laws imply rest and:

$$e = \circlearrowright$$

$$\circlearrowleft^{-1} = \circlearrowright \qquad \circlearrowright^{-1} = \circlearrowright$$

Basic group theory

Group action $\langle G, A \rangle$

Group $G = \left\langle \underline{G}, (*), e, (-)^{-1} \right\rangle$

G-**action** $A = \langle \underline{A}, \overset{A}{\circledast} \rangle$:

$\underline{A}$(**carrier** set) $\qquad \dfrac{g : \underline{G} \qquad x : \underline{A}}{g \circledast x : \underline{A}}$(**action** of G on $\underline{A}$)

such that:

$$\mathsf{g}, \mathsf{h} : \underline{G}, \mathsf{x} : \underline{A} \vdash \mathsf{g} \circledast (\mathsf{h} \circledast \mathsf{x}) = (\mathsf{g} * \mathsf{h}) \circledast \mathsf{x} \quad \text{(associativity)}$$

$$\mathsf{x} : \underline{A} \vdash e \circledast \mathsf{x} = \mathsf{x} \quad \text{(neutrality)}$$

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acting on

$\underline{\mathsf{Dir}} := \{\Leftarrow, \Rightarrow, \Uparrow, \Downarrow\}$

$\curvearrowleft \circledast \Leftarrow := \Downarrow \quad \curvearrowleft \circledast \Downarrow := \Rightarrow$

$\curvearrowleft \circledast \Rightarrow := \Uparrow \quad \curvearrowleft \circledast \Uparrow := \Leftarrow$

laws determine the rest

Basic group theory

Group action $\langle G, A \rangle$

Group $G = \left\langle \underline{G}, (*), e, (-)^{-1} \right\rangle$

G-**action** $A = \langle \underline{A}, \overset{A}{\circledast} \rangle$

Notation

- **symmetries**: $\underline{G}$ elements
- x **symmetric to** y **via** g when: $g \circledast x = y$

action and group axioms mean:

'**being symmetric**' is a
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Rotations $\langle \overset{G:=}{\mathbf{C}_4}, \overset{=:A}{\mathrm{Dir}} \rangle$

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laws determine the rest

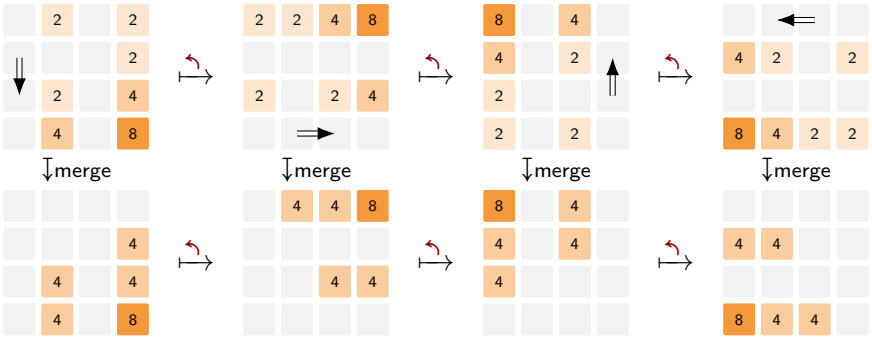
Group theory basics

Equivariant map $\varphi : A \circledast\!\!\longmapsto B$ between G-actions A, B

$$\underset{\text{function}}{\varphi : \underline{A} \to \underline{B}} \qquad \mathsf{g} : \underline{G}, \mathsf{x} : \underline{A} \vdash \varphi(\mathsf{g} \circledast \mathsf{x}) = \mathsf{g} \circledast (\varphi \mathsf{x}) : \underline{B}$$

$$\text{i.e.:} \quad \frac{x, \quad y \text{ symmetric via } g}{f\,x, f\,y \text{ symmetric via } g}$$

Example: merge : Dir $\times$ Board$_n$ $\circledast\!\!\longmapsto$ Board$_n$



w.l.o.g. anatomy: canonisation

w.l.o.g. = (input & output **symmetries**) +(**canonising** symmetries) +(**core** construction)

Canoniser for G-action A

function $c : \underline{A} \to \underline{G}$ assigning **canonising symmetries** to A-elements.

Canonical elements

$$\mathsf{Can}\, c := \{\, (c\, a) \circledast a \mid a \in \underline{A} \,\}$$

Example: mkLft : $\underline{\mathsf{Dir}} \to \underline{\mathbf{C}_4}$

Canonising symmetry makes direction 'left': mkLft $\Rightarrow$:= ↝ mkLft $\Uparrow$:= ↖ mkLft $\Leftarrow$:= ↺ mkLft $\Downarrow$:= ↘

Only canonical direction is 'left': $\mathsf{Can}\,\mathsf{mkLft} = \{\Leftarrow\}$

w.l.o.g. anatomy: core

w.l.o.g. = (input & output **symmetries**)+(**canonising** symmetries) +(**core** construction)

Core function $f : \mathrm{Can}\, c \to \underline{B}$

work on canonical elements, typically more concrete, after eliminating symmetric cases

Example: merging left

$$
\begin{aligned}
&\mathrm{merge}^{\mathrm{row}}_{\Leftarrow} && : \mathrm{Row}_n \to \mathrm{Row}_n \\
&\mathrm{merge}^{\mathrm{row}}_{\Leftarrow}\ \langle\rangle && := \langle\rangle \\
&\mathrm{merge}^{\mathrm{row}}_{\Leftarrow}\ \langle 0\rangle +\!\!+ r && := (\mathrm{merge}^{\mathrm{row}}_{\Leftarrow}\ r) +\!\!+ \langle 0\rangle \\
&\mathrm{merge}^{\mathrm{row}}_{\Leftarrow}\ \langle x\rangle && := \langle x\rangle \\
&\mathrm{merge}^{\mathrm{row}}_{\Leftarrow}\ (\langle x\ , 0\ \rangle +\!\!+ r) && := \mathrm{merge}_{\Leftarrow}\ (\langle x\rangle +\!\!+ r) +\!\!+ \langle 0\rangle \\
&\mathrm{merge}^{\mathrm{row}}_{\Leftarrow}\ \left(\left\langle 2^k, 2^k\right\rangle +\!\!+ r\right) && := \left\langle 2^{k+1}\right\rangle +\!\!+ (\mathrm{merge}^{\mathrm{row}}_{\Leftarrow}\ r) +\!\!+ \langle 0\rangle \\
&\mathrm{merge}^{\mathrm{row}}_{\Leftarrow}\ \left(\left\langle 2^k, 2^\ell\right\rangle +\!\!+ r\right) && := \left\langle 2^k\ \right\rangle +\!\!+ \mathrm{merge}^{\mathrm{row}}_{\Leftarrow}\ \left(\left\langle 2^\ell\right\rangle +\!\!+ r\right)
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{merge}_{\Leftarrow} : \mathrm{Board}_n \to \mathrm{Board}_n \\
&\mathrm{merge}_{\Leftarrow} := \mathrm{map}\ \mathrm{merge}^{\mathrm{row}}_{\Leftarrow}
\end{aligned}
$$

Symmetric programming **abstraction**

w.l.o.g. = (input & output **symmetries**)+(**canonising** symmetries)+(**core** construction)

W.l.o.g. construction

Formally:

$$\frac{G\text{-actions } A, B \qquad c : \underline{A} \to \underline{G} \qquad f : \mathsf{Can}\, c \to \underline{B}}{\mathsf{wlog}(c, f) \coloneqq \left(a \mapsto (ca)^{-1} \circledast f((ca) \circledast a)\right) : \underline{A} \to \underline{B}}$$

Example

```
mergeUnchecked : {n : Nat} -> Dir -> Board n -> Board n
mergeUnchecked = Prelude.curry $ wlog
        (\(dir, _) => makeLeft dir)                    -- canoniser
        (\case (Lt, board) => mergeLeft board          -- core function
               _ => believe_me "oops")                 --   unreachable case
```

Symmetric programming **abstraction**

w.l.o.g. = (input & output **symmetries**)+(**canonising** symmetries)+(**core** construction)

W.l.o.g. construction

Formally:

$$\frac{G\text{-actions } A, B \qquad c : \underline{A} \to \underline{G} \qquad f : \mathsf{Can}\, c \to \underline{B}}{\mathsf{wlog}(c, f) \coloneqq \left(a \mapsto (ca)^{-1} \circledast f((ca) \circledast a)\right) : \underline{A} \to \underline{B}}$$

Example

```
core : {n : Nat} -> Canon (Canoniser {n}) -> Board n
core  (( d, b) ** ev) with 0 (canonicalView ev)  -- McBride-McKinna view
 core ((Lt, b) ** ev) | Refl = mergeLeft b

merge : {n : Nat} -> Dir -> Board n -> Board n
merge = Prelude.curry $ wlog' Canoniser core
```

Equivariant where defined core functions

Assume an equivariant $\varphi : A \circledast\!\!\rightarrow B$ extends a core function $f : \mathsf{Can}\, c \to \underline{B}$. Then:

$$\begin{array}{ccccc} & \varphi \text{ extends } f & & \varphi \text{ is equivariant} & & \varphi \text{ extends } f & \\ & \downarrow & & \downarrow & & \downarrow & \\ f(g \circledast x) & = & \varphi(g \circledast x) & = & g \circledast \varphi(x) & = & g \circledast fx \end{array}$$

Equivariant where defined core functions

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Let A, B be G-actions and $X \subseteq \underline{A}$. Then, $f : X \to \underline{B}$ is **equivariant** when

$$\forall g \in \underline{G}. \forall x \in X. (g \overset{A}{\circledast} x \in X \implies f(g \overset{A}{\circledast} x) = g \overset{B}{\circledast} fx)$$

Equivariant where defined core functions

Assume an equivariant $\varphi : A \circledast\!\!\rightarrow B$ extends a core function $f : \mathsf{Can}\, c \to \underline{B}$. Then:

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Theorem

Given:
G-actions A, B; canoniser $c : \underline{A} \to \underline{G}$; equivariant core function $f : (\mathsf{Can}\, c \subseteq A) \circledast\!\!\rightarrow B$;
Then $\mathsf{wlog}(c, f) : A \circledast\!\!\rightarrow B$ *is the unique equivariant function extending f.*

Symmetric programming

Simply-typed w.l.o.g.

- Straightforward to implement
- Straightforward to use
- Plumbing group actions with type-classes/interfaces fiddly as usual
- Plenty of unreachable cases
- Doesn't cover mathematical reasoning

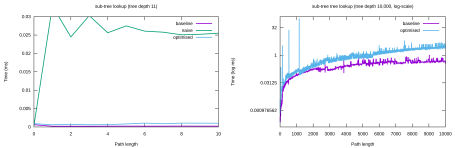
Straightforward extension to dependent types via indexed families.

Further contributions

- Dependent-types
- Data-types (simply-typed + dependently-typed)

simply-typed concept	dependently-typed concept	categorical concept
$\mathbf{Set}$	$\mathbf{Fam}_{\underline{\Gamma}}$	category $\mathcal{C}$
functor $F : \mathbf{Set} \to \mathbf{Set}$	functor $F : \mathbf{Fam}_{\underline{\Gamma}} \to \mathbf{Fam}_{\underline{\Gamma}}$	functor $F : \mathcal{C} \to \mathcal{C}$
free G-action monad	free Γ-indexed G-action monad	monad structure T over
$\underline{T}X \coloneqq \underline{G} \times X$	$\underline{T}(\underline{\Gamma} \vdash X) \coloneqq \underline{\Gamma} \vdash G \boxtimes X$	$\underline{T} : \mathcal{C} \to \mathcal{C}$
G-action	Γ-indexed G-action	T-algebra
G-strength st : $\underline{G} \times FX \to F(\underline{G} \times X)$	Γ-indexed G-strength $\mathrm{st} : G \boxtimes (FX) \to F(G \boxtimes X)$	monad-functor distributive law $\mathrm{st} : TF \to FT$
functor $S\underline{X} \coloneqq \underline{G} \to X$	functor $S(\underline{\Gamma} \vdash X) \coloneqq \underline{\Gamma} \vdash G \multimap X$	right-adjoint $\underline{T} \dashv S$
parameterised traversal $\underline{G} \times \boldsymbol{\mu} F \xrightarrow{\mathbf{pfold}\left(F X \xrightarrow{f} X\right)} X$	indexed parameterised traversal $G \boxtimes \boldsymbol{\mu} F \xrightarrow{\mathbf{pfold}\left(F X \xrightarrow{f} X\right)} X$	parameterised fold $T \boldsymbol{\mu} F \xrightarrow{\mathbf{pfold}\left(F X \xrightarrow{f} X\right)} X$
equivariant F-algebra $f : FA \looparrowright A$	indexed equivariant F-algebra $f : FA \looparrowright A$	F-algebra for lifted functor $F : \mathcal{C}^T \to \mathcal{C}^T$
canoniser $c : F \to \underline{G}$	indexed canoniser $c : F \to \underline{G}$	
canonical elements $\mathrm{Can}\, c : G\text{-}\mathbf{Act} \to \mathbf{Set}$	canonical elements $\mathrm{Can}\, c : G\text{-}\mathbf{Act}_{\Gamma} \to \mathbf{Fam}_{\underline{\Gamma}}$	canonical elements $\mathrm{Can}\, c : \mathcal{C}^T \to \mathcal{C}$

- Performance engineering and prototypes (Lean and Idris 2):



Road-map

Future work

- more examples:
  - Harrison's w.l.o.g. tactics
  - McBride's binary search trees
  - Steinitz exchange lemma
- connections: nominal techniques and symmetry breaking
- philosophy and history of mathematics:
  - comparison with vernacular and mechanised use
  - comparison with Dijkstra's wlog principle
- implementation:
  - ergonomic abstractions
  - systematise performance engineering

Contribution

Symmetric programming

Exploit symmetries for computation and reasoning—**avoid symmetric cases**.
Mathematical foundations for symmetric programming semantics:

- **Abstractions** for symmetric programming
- **Simply-typed** reasoning about equivariance
- **Dependently-typed** and mathematical reasoning
- User-defined **data-types** through initial algebra semantics and induction principles
  - Simply-typed data-types
  - Dependently-typed/indexed data-types