Algebraic Foundations for Effect-Dependent Optimisations

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Gifford-style Types and Effects

Effect systems

$$\textbf{if true then } x := 1$$
$$\textbf{else } x := \textbf{deref}(y)$$

Gifford-style Types and Effects

Effect systems

$$\vdash \textbf{if true then } \mathtt{x} := 1 \quad \textbf{else } \mathtt{x} := \textbf{deref}(\mathtt{y}) : () \,!\, \underbrace{\{\mathtt{lookup}, \mathtt{update}\}}_{\varepsilon}$$

$$\Gamma \vdash M_i : A_i \,!\, \varepsilon_i$$

Effect-dependent optimisations [Benton et al.]

$$\varepsilon_i \subseteq \{\mathtt{lookup}\} \implies \begin{array}{c} \mathtt{let}\ x = M_1\ \mathtt{in}\ (\mathtt{let}\ y = M_2\ \mathtt{in}\ N) \\ = \\ \mathtt{let}\ y = M_2\ \mathtt{in}\ (\mathtt{let}\ x = M_1\ \mathtt{in}\ N) \end{array}$$

Problem

Difficulty

Change language or effects $\implies$ restart from scratch (useful craft).

- Duplicated effort
- Mix routine and important issues

Solution

General semantic account of effect type systems (science).

Prospect

Tools, methods and automatic support (engineering).

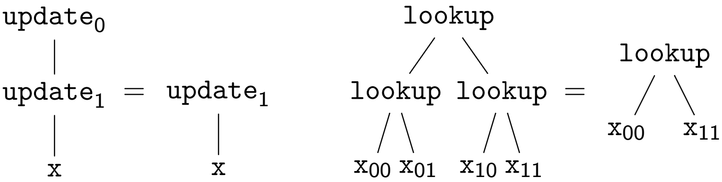
Plan

Tool: algebraic theory of effects

An interface to effects:

Effect operations Σ e.g.: `lookup` : 2, `update` : 1&2

Effect equations E e.g.:



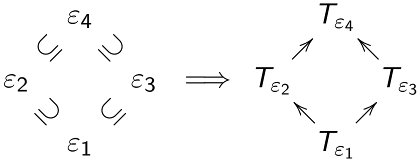
Marriage of effects and monads

Observation [Wadler]

Change notation:

$$\Gamma \vdash M : A \,!\, \varepsilon \implies \Gamma \vdash M : T_{\varepsilon} A$$

T_{ε} behaves like a monad.



Annotation effects as effect operations

Key Observation

$\varepsilon = \{\texttt{lookup}, \texttt{update}\}$ as an algebraic **signature**.

Change Perspective

View T_ε as $\langle \Sigma_\varepsilon, E_\varepsilon \rangle$

Choose $\Sigma_\varepsilon = \varepsilon$

Wonder $E_\varepsilon = ?$

Bird's Eye

monadic source language

$\Downarrow$

multi-monadic intermediate language

$\Downarrow$

semantics

$\Downarrow$

optimisations (logic)

Talk Structure

- Source language
- Intermediate representation (IR) language
- Semantics
  - Validating optimisations
  - Constructing IR models
- Optimisations
- Conclusions

Bird's Eye

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Source syntax

Signature

$\Sigma = \{op : a\&p\}$ parametrises the language.

Example

State: $\texttt{lookup} : 2\ (\texttt{lookup} : 2\&1)$, $\quad \texttt{update} : 1\&2$

Exceptions: $\texttt{DivideByZero} : 0$

Input: $\texttt{input} : 128$, $\texttt{output} : 1\&128$

Already $\mathbf{2^5} = \mathbf{32}$ different languages!

Source syntax

Types and terms

$$
\begin{array}{rcl}
A, B, \ldots & ::= & \mathbf{n} \mid A \to B \mid TA \\
M, N, \ldots & ::= & x \mid i \mid \lambda x.M \mid M\,N \\
& & \mid \mathtt{return}\, M \mid x \leftarrow M; N \\
& & \mid \mathtt{op}_M N
\end{array}
$$

Source syntax

Type system

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M} \qquad \frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\,N : B}$$

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathtt{return}\, M : TA} \qquad \frac{\Gamma \vdash M : TA \qquad \Gamma, x : A \vdash N : TB}{\Gamma \vdash x \leftarrow M; N : TB}$$

(cont.)

Source syntax

Type system

(contd.)

$$\frac{\Gamma \vdash M : \mathbf{p} \qquad \Gamma \vdash N : \mathbf{a} \to TB}{\Gamma \vdash \mathtt{op}_M N : TB}\ \mathtt{op} : a\&p$$

Example

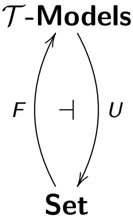
$$\vdash \mathtt{lookup}(\lambda x.\mathtt{update}_0(\lambda\_.\,\mathtt{return}\,x)) : T\mathbf{2}$$

Source semantics

Eilenberg-Moore adjunction

A model is a theory $\mathcal{T} = \langle \Sigma, E \rangle$

Derive an adjunction $F \dashv U$:



Derive a strong monad $Tx := UFx$

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optimisations (logic)

IR syntax

Types and terms

$$
\begin{array}{rcll}
A, B, \ldots & ::= & \mathbf{n} \mid A \to B & \mid T_\varepsilon A \\
M, N, \ldots & ::= & x \mid i \mid \lambda x.M & \mid M\, N \\
 & & \mid \texttt{return}_\varepsilon\, M & \mid x \leftarrow M; N \\
 & & \mid \texttt{op}_M N & \mid \texttt{coerce}_{\varepsilon \subseteq \varepsilon'} M
\end{array}
$$

where $\varepsilon, \varepsilon' \subseteq \Sigma$

Type system

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M} \qquad \frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\,N : B}$$

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathtt{return}_\varepsilon\, M : T_\varepsilon A} \qquad \frac{\Gamma \vdash M : T_\varepsilon A \qquad \Gamma, x : A \vdash N : T_\varepsilon B}{\Gamma \vdash x \leftarrow M; N : T_\varepsilon B}$$

(cont.)

Type system

(contd.)

$$\frac{\Gamma \vdash M : \mathbf{p} \qquad \Gamma \vdash N : \mathbf{a} \to T_\varepsilon B}{\Gamma \vdash \mathtt{op}_M N : T_\varepsilon B}\ \mathtt{op} : a\&p, \mathtt{op} \in \varepsilon$$

$$\frac{\Gamma \vdash M : T_\varepsilon A}{\Gamma \vdash \mathtt{coerce}_{\varepsilon \subseteq \varepsilon'} M : T_{\varepsilon'} A}$$

Example: higher-order coercion

$$\vdash \lambda f.\lambda x.\mathtt{coerce}_{\varepsilon_2 \subseteq \varepsilon_2'}(f(\mathtt{coerce}_{\varepsilon_1 \subseteq \varepsilon_1'} x)) \\ : (T_{\varepsilon_1'} A \to T_{\varepsilon_2} B) \to (T_{\varepsilon_1} A \to T_{\varepsilon_2'} B)$$

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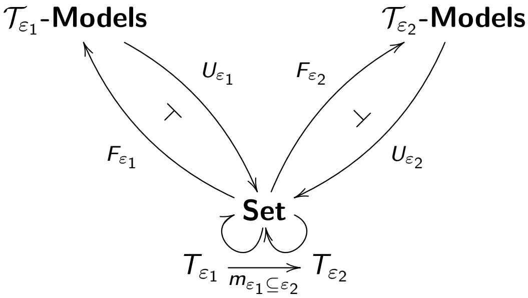
optimisations (logic)

Semantics

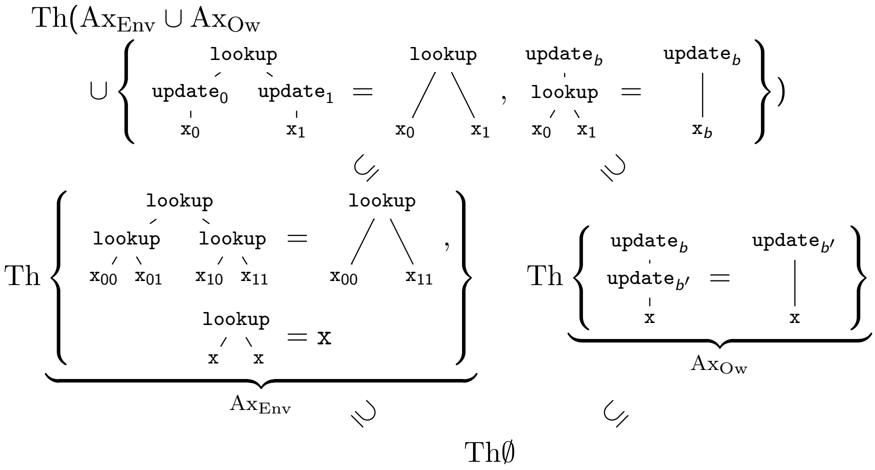
Models

A functorial family of theories: $\mathcal{T}_\varepsilon = \langle \varepsilon, E_\varepsilon \rangle$
with $E_{\varepsilon_1} \subseteq E_{\varepsilon_2}$ whenever $\varepsilon_1 \subseteq \varepsilon_2$.

Derived monads



Example


Th(Ax_Env ∪ Ax_Ow
lookup
update_0
update_1
x_0
x_1
update_b
x_b
Th
lookup
x_00
x_01
x_10
x_11
x
Ax_Env
update_b'
Ax_Ow
Th∅

Derived monads

$$T_{\{\texttt{lookup},\texttt{update}\}}A = (2 \times A)^2$$

$$T_{\{\texttt{lookup}\}}A = A^2 \qquad T_{\{\texttt{update}\}}A = (1 + 2) \times A$$

$$T_{\emptyset}A = A$$

Effect-dependent optimisation

Source: $\mathtt{x} \leftarrow M; \mathtt{return}\, 0 \quad : T\mathbf{1}$

Effect-dependent optimisation

Source: $\mathtt{x} \leftarrow M; \mathtt{return}\, 0 \quad = \quad \mathtt{return}\, 0 \quad : T\mathbf{1}$

IR:

$$\begin{aligned} &\mathtt{x} \leftarrow M; \\ &\mathtt{return}_{\{\mathtt{lookup}\}}\, 0 \quad = \quad \mathtt{return}_{\{\mathtt{lookup}\}}\, 0 \quad : T_{\{\mathtt{lookup}\}}\mathbf{1} \end{aligned}$$

crucial step holds $\forall N : T_{\{\mathtt{lookup}\}}A$, not $\forall N : TA$

Effect-dependent optimisation

Source: $\mathtt{x} \leftarrow M; \mathtt{return}\, 0 \quad = \quad \mathtt{return}\, 0 \quad : T\mathbf{1}$

IR:

$$\begin{aligned}\mathtt{x} \leftarrow M;& \\ \mathtt{return}_{\{\mathtt{lookup}\}}\, 0 \quad &= \quad \mathtt{return}_{\{\mathtt{lookup}\}}\, 0 \quad : T_{\{\mathtt{lookup}\}}\mathbf{1} \\ & \quad\quad \mathtt{return}_{\emptyset}\, 0 \quad : T_{\emptyset}\mathbf{1}\end{aligned}$$

Formalising soundness

Erasure

Erase : IR terms $\to$ source terms

$\mathrm{Erase}(M)$: remove ε's and coercions from M

$$\mathtt{coerce}_{\{\mathtt{lookup}\}}(\mathtt{x} \leftarrow M; \mathtt{return}_{\emptyset}\, 0) \quad \xmapsto{\mathrm{Erase}} \quad \mathtt{x} \leftarrow \mathrm{Erase}(M); \mathtt{return}\, 0$$

Validity

$\mathcal{M}$ a model (source or IR):

$$\mathcal{M} \models M = N \overset{def}{\iff} [\![M]\!] = [\![N]\!] \text{ in } \mathcal{M}$$

Formal soundness

Soundness

For a **source** model $\mathcal{T}$ and IRs $\vdash M, N : T_\varepsilon \mathbf{n}$,
suffices to find an IR model $\mathcal{T}^\sharp$ such that:

$$\mathcal{T}^\sharp \models M = N \implies \mathcal{T} \models \mathrm{Erase}(M) = \mathrm{Erase}(N)$$

Source: $\mathit{Erase}(M)$ $\mathit{Erase}(N) \quad : T\mathbf{n}$

IR: $M = M' = M'' = \ldots = M''' = N \quad : T_\varepsilon \mathbf{n}$

Constructing IR Models

Conservative Restriction Model

Given $\mathcal{T} = \langle \Sigma, E \rangle$, define the IR model $\mathcal{T}^{\mathrm{Cns}}$ by:

$$E|_{\varepsilon} := E \cap (\varepsilon\text{-terms} \times \varepsilon\text{-terms})$$

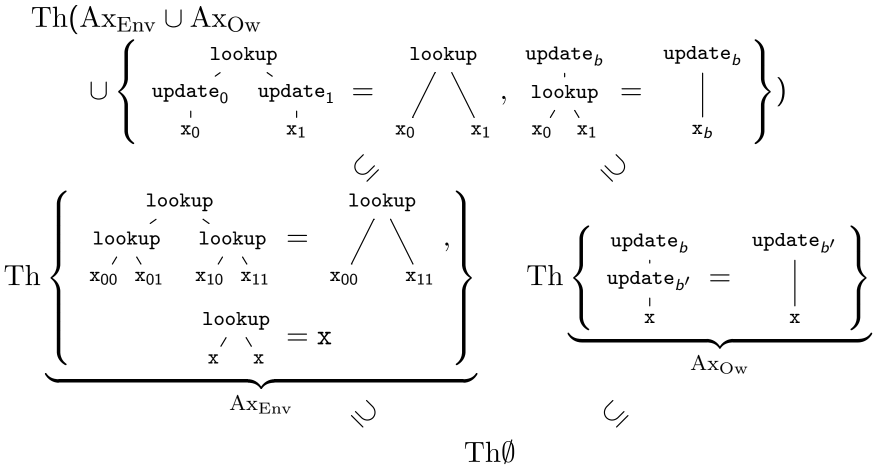
i.e., all derivable E equations between ε-terms.

Theorem

For all $\vdash M, N : T_{\varepsilon}\mathbf{n}$*:*

$$\mathcal{T}^{\mathrm{Cns}} \models M = N \iff \mathcal{T} \models \mathrm{Erase}(M) = \mathrm{Erase}(N)$$

Example Conservative Restriction Model



Global state

Injective monad morphisms

$$T_{\{\texttt{lookup},\texttt{update}\}}A = (2 \times A)^2$$

$$T_{\{\texttt{lookup}\}}A = A^2 \qquad T_{\{\texttt{update}\}}A = (1 + 2) \times A$$

$$T_{\emptyset}A = A$$

Combining theories

Sum

$\mathcal{T}^1 + \mathcal{T}^2 \coloneqq \langle \Sigma_1 + \Sigma_2, \mathrm{Th}(E_1 + E_2) \rangle$

Theorem [Hyland, Plotkin, Power]

Summing with exceptions (resp. input, output) induces the exception (resp. input, output) monad transformer.

Modularity theorem

Idea

Restrictions of $\mathcal{T} = \mathcal{T}^1 \bigcirc \mathcal{T}^2$ in terms of component restrictions.

Theorem

For consistent theories:

$$(\mathcal{T}^1 + \mathcal{T}^2)\big|_{\varepsilon_1 + \varepsilon_2} = \mathcal{T}^1\big|_{\varepsilon_1} + \mathcal{T}^2\big|_{\varepsilon_2}$$

Combining theories

Tensor

$\mathcal{T}^1 + \mathcal{T}^2 := \langle \Sigma_1 + \Sigma_2, \mathrm{Th}((E_1 + E_2) \cup E_{\Sigma_1 \otimes \Sigma_2}) \rangle$ where $E_{\Sigma_1 \otimes \Sigma_2}$ are

$$f(g(x_{00}, x_{01}), g(x_{10}, x_{11})) = g(f(x_{00}, x_{10}), f(x_{01}, x_{11}))$$

Theorem [Hyland, Plotkin, Power]

Tensoring with the global state (resp. environment, overwrite) theory induces the global state (resp. environment, overwrite) monad transformer.

Modularity counter example

Idea

Restrictions of $\mathcal{T} = \mathcal{T}^1 \bigcirc \mathcal{T}^2$ in terms of component restrictions.

Tensor counterexample: Eckmann-Hilton

$$\begin{aligned}(\text{Monoids} \otimes \text{Monoids})_{\{\cdot,1\}+\emptyset} &= \text{Commutative Monoids} \\ &\neq \text{Monoids}|_{\{\cdot,1\}} \otimes \text{Monoids}|_{\emptyset}\end{aligned}$$

$$(1 \cdot_1 x) \cdot_2 (y \cdot_1 1) = (1 \cdot_2 y) \cdot_1 (x \cdot_2 1) \implies x \cdot_2 y = y \cdot_1 x$$
$$(x \cdot_1 1) \cdot_2 (1 \cdot_1 y) = (x \cdot_2 1) \cdot_1 (1 \cdot_2 y) \implies x \cdot_2 y = x \cdot_1 x$$

Pragmatic Modularity Theorems

Tensoring with the global state, environment and overwrite theories is modular. Tensoring with non-determinism is non-modular over ω-CPO.

Axiomatic restriction

Axiomatic Restriction Model

Given $\mathcal{T} = \langle \Sigma, \mathrm{ThAx} \rangle$, define the IR model $\mathcal{T}^{\mathrm{Ax}}$ by:

$$\mathrm{Th}|_{\varepsilon}\,\mathrm{Ax} \coloneqq \mathrm{Th}(\mathrm{Ax} \cap (\varepsilon\text{-terms} \times \varepsilon\text{-terms}))$$

By fiat,

$$\begin{aligned}
\mathrm{Th}|_{\varepsilon_1+\varepsilon_2}\,(\mathrm{Ax}^1 + \mathrm{Ax}^2) &= \mathrm{Th}|_{\varepsilon_1}\,\mathrm{Ax}^1 + \mathrm{Th}|_{\varepsilon_2}\,\mathrm{Ax}^2 \\
\mathrm{Th}|_{\varepsilon_1+\varepsilon_2}\,((\mathrm{Ax}^1 + \mathrm{Ax}^2) \cup E_{\Sigma_1 \otimes \Sigma_2}) &= \mathrm{Th}|_{\varepsilon_1}\,\mathrm{Ax}^1 \otimes \mathrm{Th}|_{\varepsilon_2}\,\mathrm{Ax}^2
\end{aligned}$$

Theorem

For all $\vdash M, N : T_{\varepsilon}\mathbf{n}$:

$$\mathcal{T}^{\mathrm{Ax}} \models M = N \implies \mathcal{T} \models \mathrm{Erase}(M) = \mathrm{Erase}(N)$$

Bird's Eye

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$\Downarrow$

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optimisations (logic)

Cataloguing Optimisations

For existing transformations:

- Validate
- Classify
- Generalise

Structural properties

Structural

Bread and butter of optimisation, e.g.

- β, η rules.
- Sequencing.
- Coercion, e.g.:

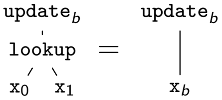
$$\texttt{coerce}_{\varepsilon' \subseteq \varepsilon''}(\texttt{coerce}_{\varepsilon \subseteq \varepsilon'} M) = \texttt{coerce}_{\varepsilon \subseteq \varepsilon''} M$$

Practically: constant propagation, common subexpression elimination, (loop unrolling), etc.

Local algebraic properties

Algebraic

Single equations in $\mathcal{T}_\varepsilon$, e.g.:



become optimisations, e.g.:

$$\texttt{update}_V(\texttt{lookup}(N)) = \texttt{update}_V N\ V$$

i.e., **local** properties of $\mathcal{T}_\varepsilon$.

Global structural properties

Discard: Utilitarian Form

$$\frac{\Gamma \vdash M : T_\varepsilon A \qquad \Gamma \vdash N : T_{\varepsilon'} B}{\mathtt{x} \leftarrow (\mathtt{coerce}_{\varepsilon \subseteq \varepsilon'} M); N = N}$$

Discard: Pristine Form

$$\frac{\Gamma \vdash M : T_\varepsilon A}{\mathtt{x} \leftarrow M; \mathtt{return}_\varepsilon\, 0 = \mathtt{return}_\varepsilon\, 0}$$

(cont.)

Abstract optimisations

(contd.) Discard: $\mathtt{x} \leftarrow M; \mathtt{return}_\varepsilon\, 0 = \mathtt{return}_\varepsilon\, 0$

Categorical Characterisation

$$T_\varepsilon 1 \cong 1$$

Due to Kock, Jacobs, Führmann

Algebraic Characterisation

For all $t(\mathtt{x}_1, \ldots, \mathtt{x}_n)$:

$$\begin{array}{ccc} & t & \\ \swarrow & & \searrow \\ \mathtt{x} & \cdots & \mathtt{x} \end{array} = \mathtt{x}$$

Due to Wraith

This is a **global** property.

Effective dictionary

Knowledge unification

Optimisation	Utilitarian	Pristine	Categorical	Algebraic
⋮	⋮	⋮	⋮	⋮

Effective dictionary

name	utilitarian form	pristine form	abstract side condition	algebraic equivalent	example basic theories
Discard	B F $\dfrac{\Gamma \vdash_{\varepsilon} M : \mathbf{F}_{\varepsilon} A \quad \Gamma \vdash_{\varepsilon'} N : \underline{B}}{(\mathbf{coerce}\, M)\ \mathbf{to}\ x : A.\, N = N}$	B F $\dfrac{\Gamma \vdash_{\varepsilon} M : \mathbf{F}_{\varepsilon} A}{M\ \mathbf{to}\ x : A.\, \mathbf{return}_{\varepsilon} \star = \mathbf{return}_{\varepsilon} \star}$	F $\mathcal{T}_{\varepsilon}$ affine: $\eta^{\varepsilon}_{\mathbb{1}} : \mathbb{1} \to \lvert F_{\varepsilon} \mathbb{1} \rvert$ has a continuous inverse	For all ε-terms t: $t(\mathbf{x}, \ldots, \mathbf{x}) = \mathbf{x}$	read-only state, convex, upper and lower semilattices
Copy	B F $\dfrac{\Gamma \vdash_{\varepsilon} M : \mathbf{F}_{\varepsilon} A \quad \Gamma, x : A, y : A \vdash_{\varepsilon'} N : \underline{B}}{\mathbf{coerce}\, M\ \mathbf{to}\ x : A.\ \mathbf{coerce}\, M\ \mathbf{to}\ y : A.\, N = \mathbf{coerce}\, M\ \mathbf{to}\ x : A.\, N[x/y]}$	B F $\dfrac{\Gamma \vdash_{\varepsilon} M : \mathbf{F}_{\varepsilon} A}{M\ \mathbf{to}\ x : A.\ M\ \mathbf{to}\ y : A.\, \mathbf{return}_{\varepsilon}(x, y) = M\ \mathbf{to}\ x : A.\, \mathbf{return}_{\varepsilon}(x, x)}$	F $\mathcal{T}_{\varepsilon}$ relevant: $\psi_{\varepsilon} \circ \delta = L^{\varepsilon} \delta$	For all ε-terms t: $t(t(\mathbf{x}_{11}, \ldots, \mathbf{x}_{1n}), \ldots, t(\mathbf{x}_{n1}, \ldots, \mathbf{x}_{nn})) = t(\mathbf{x}_{11}, \ldots, \mathbf{x}_{nn})$	exceptions, lifting, read-only state, write-only state
Weak Copy	$\dfrac{\Gamma \vdash_{\varepsilon} M : \mathbf{F}_{\varepsilon} A \quad \Gamma, x : A \vdash_{\varepsilon'} N : \underline{B}}{\mathbf{coerce}\, M\ \mathbf{to}\ x : A.\ \mathbf{coerce}\, M\ \mathbf{to}\ y : A.\, N = \mathbf{coerce}\, M\ \mathbf{to}\ x : A.\, N}$	$\dfrac{\Gamma \vdash_{\varepsilon} M : \mathbf{F}_{\varepsilon} A}{M\ \mathbf{to}\ x : A.\, M = M}$	$\mu^{\varepsilon} \circ \mathrm{str}^{\varepsilon} \circ \delta = \mathrm{id}$	For all ε-terms t: $t(t(\mathbf{x}_1, \ldots, \mathbf{x}_n), \ldots, t(\mathbf{x}_1, \ldots, \mathbf{x}_n)) = t(\mathbf{x}_1, \ldots, \mathbf{x}_n)$	any affine or relevant theory: lifting, exceptions, read-only and write-only state, all three semilattice theories
Swap	B F $\dfrac{\Gamma \vdash_{\varepsilon_1} M_1 : \mathbf{F}_{\varepsilon_1} A_1 \quad \Gamma \vdash_{\varepsilon_2} M_2 : \mathbf{F}_{\varepsilon_2} A_2 \quad \Gamma, x_1 : A_1, x_2 : A_2 \vdash_{\varepsilon'} N}{\mathbf{coerce}\, M_1\ \mathbf{to}\ x_1 : A_1.\ \mathbf{coerce}\, M_2\ \mathbf{to}\ x_2 : A_2.\, N = \mathbf{coerce}\, M_2\ \mathbf{to}\ x_2 : A_2.\ \mathbf{coerce}\, M_1\ \mathbf{to}\ x_1 : A_1.\, N}$	B F $\dfrac{\Gamma \vdash_{\varepsilon_1} M_1 : \mathbf{F}_{\varepsilon_1} A_1 \quad \Gamma \vdash_{\varepsilon_2} M_2 : \mathbf{F}_{\varepsilon_2} A_2}{\mathbf{coerce}\, M_1\ \mathbf{to}\ x_1 : A_1.\ \mathbf{coerce}\, M_2\ \mathbf{to}\ x_2 : A_2.\, \mathbf{return}_{\varepsilon}(x_1, x_2) = \mathbf{coerce}\, M_2\ \mathbf{to}\ x_2 : A_2.\ \mathbf{coerce}\, M_1\ \mathbf{to}\ x_1 : A_1.\, \mathbf{return}_{\varepsilon}(x_1, x_2)}$	F $\mathfrak{T}_{\varepsilon_1 \subseteq \varepsilon}, \mathfrak{T}_{\varepsilon_2 \subseteq \varepsilon}$ commute: $\psi_{\varepsilon} \circ (m^{\varepsilon_1 \subseteq \varepsilon} \times m^{\varepsilon_2 \subseteq \varepsilon}) = \tilde{\psi}_{\varepsilon} \circ (m^{\varepsilon_1 \subseteq \varepsilon} \times m^{\varepsilon_2 \subseteq \varepsilon})$	$\mathfrak{T}_{\varepsilon_1 \subseteq \varepsilon}$ translations commute with $\mathfrak{T}_{\varepsilon_2 \subseteq \varepsilon}$ translations (see tensor equations)	$\mathcal{T}_1 \to \mathcal{T}_1 \otimes \mathcal{T}_2 \leftarrow \mathcal{T}_2$, e.g., distinct global memory cells
Weak Swap	$\dfrac{\Gamma \vdash_{\varepsilon_1} M_1 : \mathbf{F}_{\varepsilon_1} A_1 \quad \Gamma \vdash_{\varepsilon_2} M_2 : \mathbf{F}_{\varepsilon_2} A_2 \quad \Gamma, x_1 : A_1 \vdash_{\varepsilon'} N}{\text{(same as Swap)}}$	$\dfrac{\Gamma \vdash_{\varepsilon_1} M_1 : \mathbf{F}_{\varepsilon_1} A_1 \quad \Gamma \vdash_{\varepsilon_2} M_2 : \mathbf{F}_{\varepsilon_2} A_2}{\mathbf{coerce}\, M_1\ \mathbf{to}\ x_1 : A_1.\ \mathbf{coerce}\, M_2\ \mathbf{to}\ x_2 : A_2.\, \mathbf{return}_{\varepsilon} x_1 = \mathbf{coerce}\, M_2\ \mathbf{to}\ x_2 : A_2.\ \mathbf{coerce}\, M_1\ \mathbf{to}\ x_1 : A_1.\, \mathbf{return}_{\varepsilon} x_1}$	$\psi_{\varepsilon} \circ (m^{\varepsilon_1} \times m^{\varepsilon_2}) \circ (\mathrm{id} \times \eta^{\varepsilon_2}_{\mathbb{1}}) = \tilde{\psi}_{\varepsilon} \circ (m^{\varepsilon_1} \times m^{\varepsilon_2}) \circ (\mathrm{id} \times \eta^{\varepsilon_2}_{\mathbb{1}})$	For all ε-terms $t = \mathfrak{T}_1(t')$, $s = \mathfrak{T}_2(s')$: $t(s(\mathbf{x}_1, \ldots, \mathbf{x}_1), \ldots, s(\mathbf{x}_n, \ldots, \mathbf{x}_n)) = s(t(\mathbf{x}_1, \ldots, \mathbf{x}_n), \ldots, t(\mathbf{x}_1, \ldots, \mathbf{x}_n))$	when $\mathcal{T}_{\varepsilon_2}$ is affine, e.g.: read-only state and convex, upper and lower semilattices.
Isolated Swap	$\dfrac{\Gamma \vdash_{\varepsilon_1} M_1 : \mathbf{F}_{\varepsilon_1} A_1 \quad \Gamma \vdash_{\varepsilon_2} M_2 : \mathbf{F}_{\varepsilon_2} A_2 \quad \Gamma \vdash_{\varepsilon'} N}{\text{(same as Swap)}}$	$\dfrac{\Gamma \vdash_{\varepsilon_1} M_1 : \mathbf{F}_{\varepsilon_1} A_1 \quad \Gamma \vdash_{\varepsilon_2} M_2 : \mathbf{F}_{\varepsilon_2} A_2}{\mathbf{coerce}\, M_1\ \mathbf{to}\ x_1 : A_1.\ \mathbf{coerce}\, M_2\ \mathbf{to}\ x_2 : A_2.\, \mathbf{return}_{\varepsilon} \star = \mathbf{coerce}\, M_2\ \mathbf{to}\ x_2 : A_2.\ \mathbf{coerce}\, M_1\ \mathbf{to}\ x_1 : A_1.\, \mathbf{return}_{\varepsilon} \star}$	$\psi_{\varepsilon} \circ (m^{\varepsilon_1} \times m^{\varepsilon_2}) \circ (\eta^{\varepsilon_1}_{\mathbb{1}} \times \eta^{\varepsilon_2}_{\mathbb{1}}) = \tilde{\psi}_{\varepsilon} \circ (m^{\varepsilon_1} \times m^{\varepsilon_2}) \circ (\eta^{\varepsilon_1}_{\mathbb{1}} \times \eta^{\varepsilon_2}_{\mathbb{1}})$	For all ε-terms $t = \mathfrak{T}_1(t')$, $s = \mathfrak{T}_2(s')$: $t(s(\mathbf{x}, \ldots, \mathbf{x}), \ldots, s(\mathbf{x}, \ldots, \mathbf{x})) = s(t(\mathbf{x}, \ldots, \mathbf{x}), \ldots, t(\mathbf{x}, \ldots, \mathbf{x}))$	when $\mathcal{T}\varepsilon_1$ is affine: read-only state and convex, upper and lower semilattices.
Unique	B $\dfrac{\Gamma \vdash_{\varepsilon} M_i : \mathbf{F}_{\varepsilon} \mathbf{0}, i = 1, 2}{M_1 = M_2}$	(same as utilitarian form)	$F_{\varepsilon} \mathbb{0} = \mathbb{0}, \mathbb{1}$	$\mathcal{T}_{\varepsilon}$ equates all ε-constants	all three state theories, all three semilattice theories, a single unparameterised exception, lifting
Pure Hoist	B F $\dfrac{\Gamma \vdash_{\varepsilon} M : \mathbf{F}_{\varepsilon} A \quad \Gamma, x : A \vdash_{\varepsilon'} N : \underline{B}}{\mathbf{return}_{\varepsilon}\ \mathbf{thunk}\ (\mathbf{coerce}\, M\ \mathbf{to}\ x : A.\, N) = M\ \mathbf{to}\ x : A.\, \mathbf{return}_{\varepsilon}\ \mathbf{thunk}\ N}$	B F $\dfrac{\Gamma \vdash_{\varepsilon} M : \mathbf{F}_{\varepsilon} A}{\mathbf{return}_{\varepsilon}\ \mathbf{thunk}\ M = M\ \mathbf{to}\ x : A.\, \mathbf{return}_{\varepsilon}\ \mathbf{thunk}\ \mathbf{return}_{\varepsilon} x}$	F $L^{\varepsilon} \eta^{\varepsilon}_{W} = \eta^{\varepsilon}_{\lvert F_{\varepsilon} W \rvert}$	all ε-terms are equal to variables in $\mathcal{T}_{\varepsilon}$	the empty theory, inconsistent theories
Hoist	B F $\dfrac{\Gamma \vdash_{\varepsilon} M : \mathbf{F}_{\varepsilon} A \quad \Gamma, x : A \vdash_{\varepsilon'} N : \underline{B}}{M\ \mathbf{to}\ x : A.\ \mathbf{return}_{\varepsilon}\ \mathbf{thunk}\ (\mathbf{coerce}\, M\ \mathbf{to}\ x : A.\, N) = M\ \mathbf{to}\ x : A.\, \mathbf{return}_{\varepsilon}\ \mathbf{thunk}\ N}$	B F $\dfrac{\Gamma \vdash_{\varepsilon} M : \mathbf{F}_{\varepsilon} A}{M\ \mathbf{to}\ x : A.\ \mathbf{thunk}\ \mathbf{return}_{\varepsilon}(x, \mathbf{thunk}\ M) = M\ \mathbf{to}\ x : A.\ \mathbf{thunk}\ \mathbf{return}_{\varepsilon}(x, \mathbf{thunk}\ \mathbf{return}_{\varepsilon} x)}$	F $L^{\varepsilon} \langle \eta^{\varepsilon}, \mathrm{id} \rangle = \mathrm{str}^{\varepsilon} \circ \delta$	all ε-terms are either a variable or independent of their variables via $\mathcal{T}_{\varepsilon}$	all theories containing only constants: lifting and exceptions

Figure 7. Abstract Optimisations

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2011/11/16

Further reading

Teasers

Details in the paper, and:

- An extended example:

$$\begin{aligned}\text{Exceptions} + (\text{Read Only} \otimes \text{Write Only} \otimes \text{Read-Write} \otimes \\ (\text{Exceptions} + \text{Input} + \text{Output} + \\ (\text{Non-determinism} \otimes \text{Lifting})))\end{aligned}$$

  ($2^9 = 512$ effect sets).
- Modular validation of optimisations.
- More expressible language (recursion + CBPV).
- More optimisations.
- Further work.

Summary

Conclusions

- This work unified and generalised existing work: a step towards a science and an engineering discipline.
- The algebraic approach is fruitful: clarifies and unveils both connections and constructions.
- Category theory was crucial to our formulation and for forming the connections between the different areas that were unified.

Some further work

- Effect reconstruction
- Handlers
- Automation
- More effects
- Locality
- Concurrency
- Better program logics (Hoare, modal, etc.).