A domain theory for statistical probabilistic programming

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Statistical probabilistic programming

$$[\![-]\!] : \text{programs} \to \text{unnormalised distributions}$$

- Bayesian inference: compiler computes normalisation
- Continuous types: $\mathbb{R}, [0, \infty]$
- Probabilistic effects:

normally distributed sample

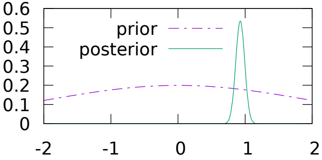
$$\mathbf{sample}(\mu, \sigma) : \mathbb{R}$$

scale distribution by r

$$\frac{r : [0, \infty]}{\mathbf{score}(r) : \mathbf{1}}$$

conditioning/fitting to observed data with likelihood

$$[\![\mathbf{sample}(0, 2)]\!]$$



prior

$$\left[\!\!\left[\begin{array}{l}\mathbf{let}\ x{=}\mathbf{sample}(0{,}2)\\ \mathbf{in}\ \mathbf{score}(\mathit{normalPdf}(1.1|\ x, \tfrac{1}{4}));\\ \quad \mathbf{score}(\mathit{normalPdf}(1.9|2x, \tfrac{1}{4}));\\ \quad \mathbf{score}(\mathit{normalPdf}(2.7|3x, \tfrac{1}{4}));\\ \quad x\end{array}\right]\!\!\right]$$

posterior

Statistical probabilistic programming

Exact Bayesian inference
using disintegration
[Shan-Ramsey'17]

- Commutativity/exchangability/Fubini

$$\left[\!\!\left[\begin{array}{l}\mathbf{let}\ x = K\ \mathbf{in}\\ \mathbf{let}\ y = L\ \ \mathbf{in}\\ f(x,y)\end{array}\right]\!\!\right] = \left[\!\!\left[\begin{array}{l}\mathbf{let}\ y = L\ \ \mathbf{in}\\ \mathbf{let}\ x = K\ \mathbf{in}\\ f(x,y)\end{array}\right]\!\!\right] \qquad \begin{array}{c}\int [\![K]\!](\mathrm{d}x)\int [\![L]\!](\mathrm{d}y) f(x,y)\\ =\\ \int [\![L]\!](\mathrm{d}y)\int [\![K]\!](\mathrm{d}x) f(x,y)\end{array}$$

probabilitity distributions ✔

σ-finite distributions ✔

arbitrary distributions ✗

s-finite distributions ✔

not closed under push-forward

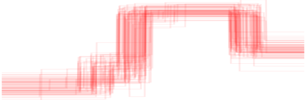
full definability [Staton'17]

Statistical probabilistic programming

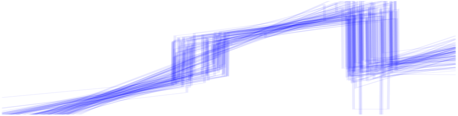
Express continuous distributions using:

- Higher-order functions:

(e.g. generative random function models)



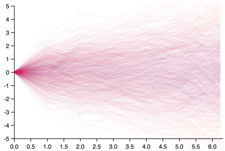
piecewise(random-constant)



piecewise(random-linear)

- Term recursion: (e.g. Gaussian random walk)

$$rw(x,\sigma) = \lambda(). \qquad \text{// thunk}$$
$$\quad \mathbf{let}\, y = \mathbf{sample}(x,\sigma)$$
$$\quad \mathbf{in}\, (x, rw\,(y,\sigma))$$



- Type recursion (à la FPC) (e.g. dynamic types, IRs)

$$Dynamic = \mu\alpha.\{\mathrm{Val}(\mathbb{R}) \mid \mathrm{Fun}(\alpha \to \alpha)\}$$

Resample-Move Sequential Monte Carlo

[Ścibior et al.'18a+b]

resamples particles moves

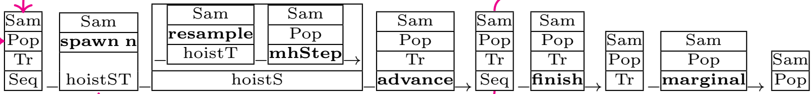
recursion

```
rmsmc k n t =
  marginal . finish . compose k (
    advance . hoistS (
      compose t mhStep . hoistT resample
    )
  ) . hoistST (spawn n >>)
```

inference representation

higher order

inference transformer



inference transformation (invariant preserving)

recursive types

ProbProg: Important Language Features

Church WebPPL Venture	sample	$\mathbb{R}$	score	higher order	term rec	type rec	Fubini (commute)
sets + probability	✓	✗	✗	✓	✗	✗	✓
meas space + subprobability	✓	✓	✗	✗	1^{st}	✗	✓
CPO + subprobability	✓	✓	✗	✓	✓	✓	?
cont domain + subprobability [Jones-Plotkin'89]	✓	✓	✗	✗	1^{st}	✗	✓
⋮ [Jung-Tix'98]	⋮	⋮	⋮	⋮	⋮	⋮	⋮
meas + s-finite distributions [Staton'17]	✓	✓	✓	✗	1^{st}	✗	✓
qbs + s-finite distributions [Heunen et al'17, Ścibior et al'18]	✓	✓	✓	✓	1^{st}	✗	✓
coh/meas cone + probability [Ehrhard-Pagani-Tasson'18, Ehrhard-Tasson'15-'19]	✓	✓ ✗	✗	✓	✓	? ✓	? ✓
ω**qbs + s-finite distributions** [This work]	✓	✓	✓	✓	✓	✓	✓

!

Summary

Contribution

- $\omega\mathbf{Qbs}$: a category of pre-domain quasi-Borel spaces
- M: commutative probabilistic powerdomain over $\omega\mathbf{Qbs}$
- Axiomatic treatment of measure and domain theory in $\omega\mathbf{Qbs}$
- Adequacy: $(\omega\mathbf{Qbs}, M)$ adequately interprets:
  - Statistical FPC
  - Untyped Statistical λ-calculus

This talk

- $\omega\mathbf{Qbs}$
- A probabilistic powerdomain
- Axiomatic treatment

Iso-recursive types: FPC

[Fiore-Plotkin'94]

type variable contexts
$\Delta = \{\alpha_1, \dots, \alpha_n\}$

$$\frac{\Delta, \alpha \vdash_{\mathrm{k}} \tau : \mathsf{type}}{\Delta \vdash_{\mathrm{k}} \mu\alpha.\tau : \mathsf{type}}$$

type recursion

$$\begin{aligned} Lam = \mu\alpha.\{&\mathrm{Bool}\{\mathrm{True} \mid \mathrm{False}\} \\ &\mid \mathrm{App}(\alpha * \alpha) \\ &\mid \mathrm{Abs}(\alpha \to \alpha)\} \end{aligned}$$

$\tau = \mu\alpha.\sigma$

$$\frac{\Gamma \vdash t : \sigma[\alpha \mapsto \tau]}{\Gamma \vdash \tau.\mathbf{roll}\,(t) : \tau} \qquad \frac{\Gamma \vdash t : \tau \qquad \Gamma, x : \sigma[\alpha \mapsto \tau] \vdash s : \rho}{\Gamma \vdash \mathbf{match}\, t\, \mathbf{with}\, \mathbf{roll}\, x \Rightarrow s : \rho}$$

Iso-recursive types: FPC

[Fiore-Plotkin'94]

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$\omega\mathbf{Cpo}$-enriched category of domains

$$[\![\Delta \vdash_{\mathrm{k}} \tau : \mathsf{type}]\!] : (\mathcal{C}^{\mathrm{op}})^n \times \mathcal{C}^n \to \mathcal{C}$$

locally continuous functor

$$[\![\Delta \vdash_{\mathrm{k}} \mu\alpha.\tau : \mathsf{type}]\!] = \text{minimal invariants}$$

[Freyd'91,92, Pitts'96]

Challenge

- probabilistic powerdomain
- commutativity/Fubini
- domain theory
- higher-order functions

continuous domains
[Jones-Plotkin'89]

open problem
[Jung-Tix'98]

traditional approach:

domain $\mapsto$ Scott-open sets $\mapsto$ Borel sets $\mapsto$ distributions/valuations

our approach:

as in
[Ehrhard-Pagani-Tasson'18]

$$\underbrace{(\text{domain}, \text{quasi-Borel space})}_{\substack{\text{separate} \\ \text{but compatible}}} \mapsto \text{distributions}$$

Rudimentary measure theory

Borel sets

- $[a, b]$ Borel
- A Borel $\implies A^{\complement}$ Borel
- $(A_n)_{n\in\mathbb{N}}$ Borel $\implies$
  $\bigcup_{n\in\mathbb{N}} A_n$ Borel

Measurable functions $f : \mathbb{R} \to \mathbb{R}$

$f^{-1}[A]$ Borel $\impliedby$ A Borel

Measures μ : Borel $\to [0, \infty]$

- monotone:
  $A \subseteq B \implies \mu(A) \leq \mu(B)$
- Scott-continuous:
  $A_0 \subseteq A_1 \subseteq \ldots \implies \mu\left(\bigcup_n A_n\right) = \bigvee_n \mu(A_n)$
- strict ($\mu\emptyset = 0$) and additive ($\mu(A \uplus B) = \mu A + \mu B$)

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Example (Lebesgue measures)

1 dimensional

$\lambda[a, b] = b - a$ on $\mathbb{R}$

$(\lambda \otimes \lambda)([a, b] \times [c, d]) =$
$(b - a)(d - c)$ on $\mathbb{R}^2$

2 dimensional

Example (Push-forward measure)

$f_*\mu(A) \coloneqq \mu\left(f^{-1}[A]\right)$

Borel set

measure

$f : \mathbb{R} \to \mathbb{R}$

Quasi-Borel pre-domains

set

partial order on $\underline{X}$

subset of functions
$\mathbb{R} \to \underline{X}$
"random elements"

$\boldsymbol{\omega}$**-qbs**:

$$X = (\underline{X}, \leq_X, M_X)$$

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ω-qbs:

$$X = (\underline{X}, \leq_X, M_X)$$

ω-cpo

- $x_0 \leq x_1 \leq x_2 \leq \ldots \qquad \Longrightarrow \qquad \exists \bigvee_n x_n$

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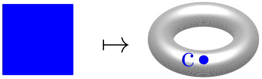
space

- $\lambda\_.x \in M_X$



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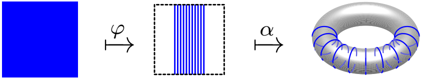
quasi-Borel space

$\mathbb{R} \xrightarrow{\varphi}_{\text{Borel}} \mathbb{R}$

- $\lambda\_.x \in M_X$
- $\alpha \in M_X \implies \alpha \circ \varphi \in M_X$



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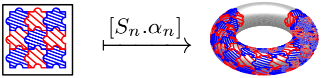

Borel measurable countable partition $\mathbb{R} = \biguplus_{n \in \mathbb{N}} S_n$

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Borel measurable
countable partition
$\mathbb{R} = \biguplus_{n \in \mathbb{N}} S_n$

s.t.:

pointwise
ω-chain

pointwise
lub

$$(\alpha_n) \in M_X^\omega \implies \bigvee_n \alpha_n \in M_X$$

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Morphisms $f : X \to Y$: Scott continuous qbs maps

monotone and
$f \bigvee_n x_n = \bigvee_n f\, x_n$

$\forall \alpha \in M_X.$
$f \circ \alpha \in M_Y$

Quasi-Borel pre-domains

$$\begin{tikzcd} \omega\mathbf{Qbs} \arrow[rr, bend left] \arrow[rr, phantom, "\top"] & & \mathbf{Qbs} \arrow[ll, bend left] \end{tikzcd}$$

Example

$S = (\underline{S}, \Sigma_S)$ measurable space

$$(\underline{S}, =, \{\alpha : \mathbb{R} \to \underline{S} | \alpha \text{ Borel measurable}\})$$

so $\mathbb{R} \in \omega\mathbf{Qbs}$

Reminder

wqbs: $X = (\underline{X}, \leq_X, M_X)$

- $\lambda\_.x \in M_X$
- $\alpha \in M_X \implies \alpha \circ \varphi \in M_X$
- $(\alpha_n \in M_X)_{n\in\mathbb{N}} \implies [r \in S_n.\alpha(r)] \in M_X$

s.t.:

$$(\alpha_n) \in M_X^\omega \qquad \implies \qquad \bigvee_n \alpha_n \in M_X$$

Quasi-Borel pre-domains

$$\omega\mathbf{Qbs} \underset{}{\overset{}{\rightleftarrows}} \omega\mathbf{Cpo} \quad \top$$

Example

$P = (\underline{P}, \leq_P)$ ω-cpo

lubs of
step functions

$$\left(\underline{P}, \leq_P, \left\{ \bigvee_k [\_ \in S_n^k . a_n^k] \,\middle|\, \forall k . \mathbb{R} = \biguplus_n S_n^k \right\}\right)$$

so $\mathbb{L} = ([0,\infty], \leq, \{\alpha : \mathbb{R} \to [0,\infty] | \alpha \text{ Borel measurable}\}) \in \omega\mathbf{Qbs}$

Reminder

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$$(\alpha_n) \in M_X^\omega \qquad \implies \qquad \bigvee_n \alpha_n \in M_X$$

Quasi-Borel pre-domains

$$\omega\mathbf{Qbs} \circlearrowright$$

Example

X ω-qbs

$$X_\perp := \left(\{\perp\} + \underline{X}, \perp \leq \underline{X}, \left\{[S.\perp, S^{\complement}.\alpha] \middle| \alpha \in M_X, S \text{ Borel}\right\}\right)$$

Reminder

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Quasi-Borel pre-domains

Products

$$\underline{X_1 \times X_2} = \underline{X}_1 \times \underline{X}_2 \qquad\qquad x \leq y \iff \forall i. x_i \leq y_i$$

$$M_{X_1 \times X_2} = \{(\alpha_1, \alpha_2) : \mathbb{R} \to \underline{X}_1 \times \underline{X}_2 | \forall i. \alpha_i \in M_{X_i}\}$$

correlated
random elements

Quasi-Borel pre-domains

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Theorem

$\omega\mathbf{Qbs} \to \omega\mathbf{Cpo} \times \mathbf{Qbs}$ *creates limits*

Quasi-Borel pre-domains

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Exponentials

- $\underline{Y^X} = \{f : \underline{X} \to \underline{Y} | f$ Scott continuous qbs morphism$\}$
  $= \mathbf{Qbs}(X, Y)$
- $f \leq g \iff \forall x \in \underline{X}. f(x) \leq g(x)$
- $M_{Y^X} = \left\{ \alpha : \mathbb{R} \to \underline{Y^X} \middle| \begin{array}{l} \text{uncurry}\, \alpha : \mathbb{R} \times X \to Y \\ \text{Scott continuous qbs morphism} \end{array} \right\}$

so $\underline{Y}^{\mathbb{R}} = M_Y$

Fundamentals of measure theory

s-finite measures

- μ_n **bounded**: $\mu_n(\mathbb{R}) < \infty$
- μ **s-finite**: $\mu = \sum_n \mu_n$, μ_n bounded

Randomisation Theorem

Every s-finite measure is a push-forward of Lebesgue:

$$\mu \text{ s-finite} \implies \mu = f_*\lambda \text{ for some } f : \mathbb{R} \to \mathbb{R}_\perp$$

Transfer principle

$\tau_*\lambda = \lambda \otimes \lambda$ for some measurable $\tau : \mathbb{R} \to (\mathbb{R} \times \mathbb{R})_\perp$

Randomisation monad structure

- $(X_\perp)^{\mathbb{R}}$
- $\mathrm{return}_X(x) : r \in [0,1] \mapsto x$
- $(\alpha \gg\!= f) : \mathbb{R} \xrightarrow{\tau} \mathbb{R} \times \mathbb{R} \xrightarrow{\mathbb{R}\times\alpha} \mathbb{R} \times X \xrightarrow{\mathbb{R}\times f} \mathbb{R} \times (Y_\perp)^{\mathbb{R}} \xrightarrow{\mathrm{eval}} Y$

$\mathbb{R} \to X_\perp$ $\qquad$ $X \to (Y_\perp)^{\mathbb{R}}$

- sample from randomisation of normal distribution
- $\mathrm{score}(r) : r' \in [0, |r|] \mapsto ()$

monad laws fail (associativity)

Lebesgue integration

Lebesgue measure

$$\alpha \begin{smallmatrix}\mathbb{R}\\ \downarrow \\ X_\perp\end{smallmatrix}, f \begin{smallmatrix}X\\ \downarrow\\ \mathbb{L}\end{smallmatrix} \mapsto f\circ\alpha \begin{smallmatrix}\mathrm{Dom}(\alpha)\\ \downarrow \mathsf{Borel}\\ \mathbb{L}\end{smallmatrix} \mapsto \int_{\mathrm{Dom}(\alpha)} f\circ\alpha(x)\lambda(\mathrm{d}x)$$

Borel maps and natural order

$\mathrm{Dom}\,(\alpha)$ Borel

$$\int : (X_\perp)^{\mathbb{R}} \longrightarrow \mathbb{L}^{\mathbb{L}^X}$$

preserves monad structure

randomisation monad structure

continuation monad

A probabilistic powerdomain

$$\begin{tikzcd}
(X_\perp)^{\mathbb{R}} \arrow[rr, "\int"] \arrow[dr, two heads] & {} \arrow[d, phantom, "="] & \mathbb{L}^{\mathbb{L}^X} \\
& MX \arrow[ur, tail] &
\end{tikzcd}$$

MX: randomisable integration operators

A probabilistic powerdomain

$$\begin{tikzcd}[column sep=small] X \arrow[rr, "f"] \arrow[dr, two heads] \arrow[rr, phantom, shift right=6, "="] & & Y \\ & \left(\mathrm{Cl}_\omega f[X], \leq_Y, \mathrm{Cl}_\omega^{Y^{\mathbb{R}}} f \circ [M_X]\right) \arrow[ur, tail] & \end{tikzcd}$$

lubs of lubs of ... f images

pointwise order

$(\mathcal{E}, \mathcal{M}) :=$ (densely strong epi, full mono) factorisation system

A probabilistic powerdomain

$\mathcal{E} =$ densely strong epis closed under:

- products: $e_1, e_2 \in \mathcal{E} \implies e_1 \times e_2 \in \mathcal{E}$
- lifting: $e \in \mathcal{E} \implies e_\perp \in \mathcal{E}$
- random elements: $e \in \mathcal{E} \implies e^{\mathbb{R}} \in \mathcal{E}$

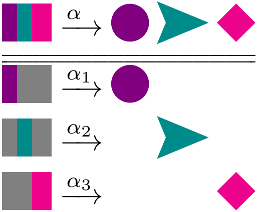
$\implies$ M strong monad for sampling + conditioning

[Kammar-McDermott'18]

A probabilistic powerdomain

$$\begin{tikzcd}
(X_\perp)^{\mathbb{R}} \arrow[rr] \arrow[dr, two heads] & \arrow[d, phantom, "="] & \mathbb{L}^{\mathbb{L}^X} \\
& MX \arrow[ur, tail] &
\end{tikzcd}$$

- M locally continuous $\implies$ may appear in domain equations
- M commutative $\implies$ satisfies Fubini
- M models synthetic measure theory
  $M \sum_{n \in \mathbb{N}} X_n \cong \prod_{n \in \mathbb{N}} M X_n$



[Kock'12,
Ścibior et al.'18]

- $MX \cong \left\{ \mu|_{\text{Scott opens}} \,\middle|\, \mu \text{ is s-finite} \right\}$ generalises valuations

standard Borel space

Axiomatic domain theory

[Fiore-Plotkin'94, Fiore'96]

Structure

- Total map category: $\omega\mathbf{Qbs}$
- Admissible monos: **Borel-open** map $m : \underset{\text{qbses}}{X} \overset{\text{strong mono}}{\rightarrowtail} \underset{\text{qbses}}{Y}$:

$$\forall \beta \in M_Y. \qquad \beta^{-1}[m[X]] \in \mathcal{B}(\mathbb{R})$$

  take Borel-Scott open maps as admissible monos

- $\mathbf{Pos}$-enrichment: pointwise order
- Pointed monad on total maps: the powerdomain

$\Longrightarrow$ model axiomatic domain theory

$\Longrightarrow$ solve recursive domain equations

Axiomatic domain theory

Structure

$\mathfrak{D}$	total map category $\omega\mathbf{Qbs}$
$f \leq g$	$\mathbf{Pos}$-enrichment pointwise order
$\mathcal{M}_{\mathfrak{D}}$	admissible monos Borel-Scott opens
T	monad for effects power-domain
m	partiality encoding $m : -_{\perp} \to T$, $\perp \mapsto \underline{0}$

Axioms

$(\dashv)$	every object has a partial map classifier $\downarrow_X : X \to X_{\perp}$
(fup)	every admissible mono is full and upper-closed
$(\dashv_{\leq})$	$\lfloor - \rfloor$ is locally monotone
$(\bigvee)$	$\mathfrak{D}$ is $\omega\mathbf{Cpo}$-enriched
(U)	ω-colimits behave uniformly
$(\mathbb{1})$	$\mathfrak{D}$ has a terminal object
$(\to_{\leq})$	$\mathfrak{D}$ has locally monotone exponentials
$(+)$	locally continuous total coproducts
$(?!)$	$\mathbb{0} \to \mathbb{1}$ is admissible
$(\times_{\bigvee})$	$\mathfrak{D}$ has a locally continuous products
(CL)	$\mathfrak{D}$ is cocomplete
$(T_{\bigvee})$	T is locally continuous

Derived axioms/structure

$\mathbf{p}\mathfrak{D}$	partial map category
$-_{\perp}$	partiality monad
$(\dashv_{\bigvee})$	the adjunction $J \dashv L$ is locally continuous
$(\mathbf{p}_{\bigvee})$	$\mathbf{p}\mathfrak{D}$ is $\omega\mathbf{Cpo}$-enriched
$(\mathbb{1}_{\leq})$	$\mathbf{p}\mathfrak{D}$ has a partial terminal
$(\otimes)$	$\mathbf{p}\mathfrak{D}$ has partial products
$(\otimes_{\bigvee})$	$(\otimes)$ is locally continuous
$(\to_{\bigvee})$	$\mathfrak{D}$ has locally continuous exponentials
$(\rightharpoonup_{\bigvee})$	$\mathbf{p}\mathfrak{D}$ has locally continuous partial exponentials
$(\mathbf{p}CL)$	$\mathbf{p}\mathfrak{D}$ is cocomplete
$(\mathbf{p}+_{\bigvee})$	$\mathbf{p}\mathfrak{D}$ has locally continuous partial coproducts
(BC)	$J : \hookrightarrow \mathbf{p}\mathfrak{D}$ is a bilimit compact expansion

Summary

Contribution

- $\omega\mathbf{Qbs}$: a category of pre-domain quasi-Borel spaces
- M: commutative probabilistic powerdomain over $\omega\mathbf{Qbs}$
- Axiomatic treatment of measure and domain theory in $\omega\mathbf{Qbs}$
- Adequacy: $(\omega\mathbf{Qbs}, M)$ adequately interprets:
  - Statistical FPC
  - Untyped Statistical λ-calculus

[Fiore-Plotkin'94, Fiore'96]

This talk

- $\omega\mathbf{Qbs}$
- A probabilistic powerdomain
- Axiomatic treatment

Also in the paper

- Axiomatic domain theory
- Operational semantics à la [Borgström et al.'16]
- Characterising $\omega\mathbf{Qbs}$

ProbProg: Important Language Features

Church WebPPL Venture	sample	$\mathbb{R}$	score	higher order	term rec	type rec	Fubini (commute)
sets + probability	✓	✗	✗	✓	✗	✗	✓
meas space + subprobability	✓	✓	✗	✗	1^{st}	✗	✓
CPO + subprobability	✓	✓	✗	✓	✓	✓	?
cont domain + subprobability [Jones-Plotkin'89]	✓	✓	✗	✗	1^{st}	✗	✓
⋮ [Jung-Tix'98]	⋮	⋮	⋮	⋮	⋮	⋮	⋮
meas + s-finite distributions [Staton'17]	✓	✓	✓	✗	1^{st}	✗	✓
qbs + s-finite distributions [Heunen et al'17, Ścibior et al'18]	✓	✓	✓	✓	1^{st}	✗	✓
coh/meas cone + probability [Ehrhard-Pagani-Tasson'18, Ehrhard-Tasson'15-'19]	✓	✓ ✗	✗	✓	✓	? ✓	? ✓
! ω**qbs + s-finite distributions** [This work]	✓	✓	✓	✓	✓	✓	✓