Mathematical Foundations for Symmetric Programming

Ohad Kammar, LFCS, University of Edinburgh
Matija Pretnar, FMF, University of Ljubljana

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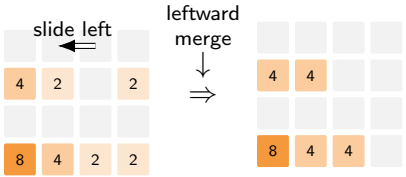
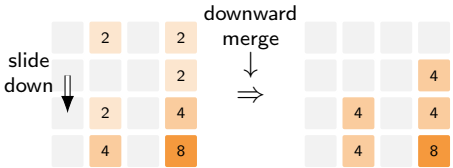
Symmetric programming

Goal: exploit **symmetry** for **computation** and **reasoning**

Example: 2048 game[1] [Cirulli'14]





[1] Available to play: `https://play2048.co`

2048 in Idris 2 [Brady'21]

Merging left goes with the grain

```
(.pad0) : Row n -> Row (1 + n)
xs.pad0 = replace {p = Row}
                  (plusCommutative _ _)
        $ xs ++ [0]
mergeRow : Row n -> Row n
mergeRow []                = []
mergeRow (     0 :: xs)    = (mergeRow xs).pad0
mergeRow (     x :: [])    = [x]
mergeRow (x :: 0 :: xs)    = (mergeRow (x :: xs)).pad0
mergeRow (x :: y :: xs) with (x == y)
                        _ | True  = 2*x :: (mergeRow xs).pad0
                        _ | False =   x :: mergeRow (y :: xs)
mergeLeft : Board n -> Board n
mergeLeft = map mergeRow
```

Merging in 2048

Going against the grain

merging **up/down** incompatible with **inductive structure**

⇓

use a different representation

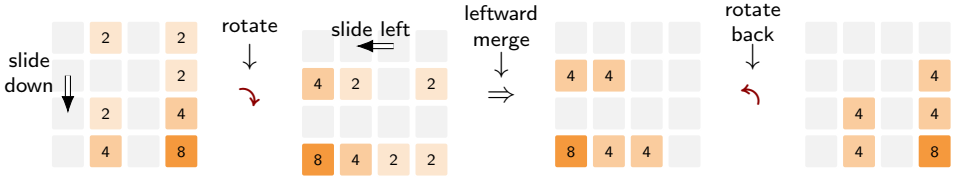
⇓

parameters galore![2]

```
  var vector     = this.getVector(direction);
  var traversals = this.buildTraversals(vector);
// ...
GameManager.prototype.getVector =
 function (direction) {// Vectors representing tile movement
  var map = { 0: { x:  0, y: -1 },  // Up
              1: { x:  1, y:  0 },  // Right
              2: { x:  0, y:  1 },  // Down
              3: { x: -1, y:  0 }}; // Left
  return map[direction];
};
```

[2] See the function `GameManager.prototype.move`:
https://github.com/gabrielecirulli/2048/blob/478b6ec346e3787f589e4af751378d06ded4cbbc/js/game_manager.js#L129.

Easy 2048 implementation using symmetry

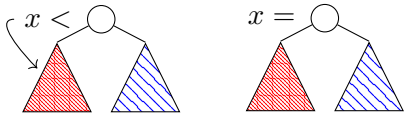


Anecdotal evidence with half-a-dozen UG students

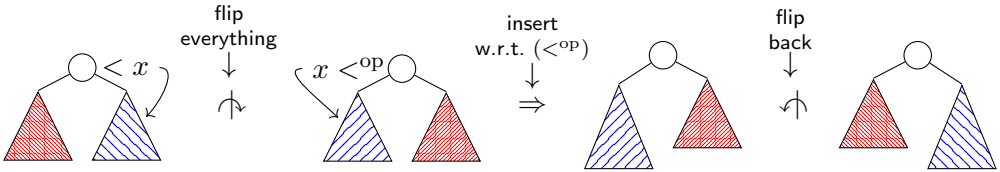
- ✗ buggy parametric solution
- ✓ mostly right symmetric solution

Example: binary search tree insertion using **symmetry**

Interesting cases



Symmetric case



NB: Yes, silly performance. Won't improve performance today.

Example: **proofs** using **symmetry** [adapted from Harrison'09]

Theorem (Schur's inequality)

Then, for all $x_1, x_2, x_3 > 0$, we have, for all t:

$$x_1^t(x_1 - x_2)(x_1 - x_3) + x_2^t(x_2 - x_3)(x_2 - x_1) + x_3^t(x_3 - x_1)(x_3 - x_2) \geq 0$$

Proof

Without loss of generality, assume $x_1 \geq x_2 \geq x_3$. Then:

$$\begin{aligned} x_1^t(x_1 - x_2)(x_1 - x_3) + x_2^t(x_2 - x_3)(x_2 - x_1) + x_3^t(x_3 - x_1)(x_3 - x_2) = \\ (x_1 - x_2)\left(x_1^t(x_1 - x_3) - x_2^t(x_2 - x_3)\right) + x_3^t(x_1 - x_3)(x_2 - x_3) \geq \\ (x_1 - x_2)(x_1^t - x_2^t)(x_1 - x_3) + x_3^t(x_1 - x_3)(x_2 - x_3) \geq 0 \end{aligned}$$

■

Example: **proofs** using **symmetry** [adapted from Harrison'09]

Theorem (Schur's inequality)

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■

Rocq SSReflect[3]

The general syntax of without loss is:

```
wlog suff? clear_switch? i_item? : ident* / term
```

On a goal `G`, it creates two subgoals: a first one to prove the formula `(term -> G) -> G` and a second one to prove the formula `term -> G`. If the optional list of ident is present on the left side of `/`, these constants are generalized in the premise `(term -> G)` of the first subgoal.

[3] Rocq Reference Manual:
https://rocq-prover.org/doc/V9.2.0/refman/proof-engine/ssreflect-proof-language.html#rocq:tacn.wlog

w.l.o.g. mechanisations

Agda[4]

```
-- Without Loss of Generality
module _  {_R_ : Rel A ℓ1} {Q : Rel A ℓ2} where

wlog : Total _R_ → Symmetric Q →
       (∀ a b → a R b → Q a b) →
       ∀ a b → Q a b
```

[4] Agda Stdlib:
https://github.com/agda/agda-stdlib/blob/c2abce1b8e72ccae49f73791d0996eaf23674357/src/Relation/Binary/Consequences.agda#L197

w.l.o.g. mechanisations

Lean4[5]

The tactic `wlog h : P` will add an assumption `h : P` to the main goal, and add a new goal that requires showing that the case `¬h : P` can be reduced to the case where **P** holds (typically by symmetry).

[5] Mathlib4 documentation:
https://leanprover-community.github.io/mathlib4_docs/Mathlib/Tactic/WLOG.html

uniform w.l.o.g. using **symmetry**

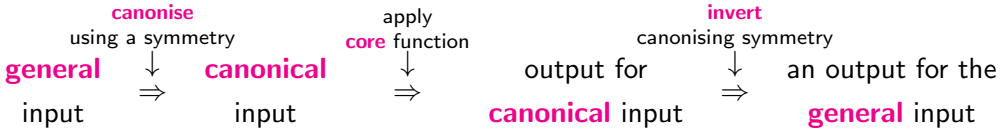
$$x_1, x_2, x_3 \in \mathbb{R}_+ \quad \overset{\text{sort using } \pi}{\Rightarrow} \quad x_{\pi 1} \geq x_{\pi 2} \geq x_{\pi 3} \quad \overset{\text{prove using additional assumption}}{\Rightarrow} \quad P(x_{\pi 1}, x_{\pi 2}, x_{\pi 3}) \quad \overset{\text{permute back since } P \text{ is invariant}}{\Rightarrow} \quad P(x_1, x_2, x_3)$$

NB: revisionism:

- ✓ Dijkstra'95 [EWD1223, not mechanised]
- ✓ HOL WLOG tactics [Harrison'09]
- ✗ Rocq SSReflect
- ✓ Agda Stdlib
- ✗ Lean

w.l.o.g. **anatomy**

w.l.o.g. = (input & output **symmetries**)+(**canonising** symmetries)+(**core** function/proof)



symmetry = **group** + **action**

Contribution

Symmetric programming

Exploit symmetries for computation and reasoning—**avoid symmetric cases**.
Mathematical foundations for symmetric programming semantics:

- **Abstractions** for symmetric programming
- **Simply-typed** reasoning about equivariance
- **Dependently-typed** and mathematical reasoning
- User-defined **data-types** through initial algebra semantics and induction principles
  - Simply-typed data-types
  - Dependently-typed/indexed data-types

Talk structure

- Introduction
- Group theory basics
- Simply-typed symmetric programming
- Dependently-typed symmetric programming
  - Indexed actions and w.l.o.g.
  - Mathematical reasoning
- Conclusion

Not today / offline

- Dijkstra'95 & Harrison'09 w.l.o.g. principles
- Symmetric user-defined data-types
  - Simply-typed data-types
  - Indexed data-types
- Equivariant reasoning

Group theory basics

I'm **not** going to do:

Group G (textbook definition)

monoid $G = \langle \underline{G}, (*), e \rangle$ with inverses:

$$g * g^{-1} = e = g^{-1} * g$$

(Non)-intuition

Only helps you if you know about monoids, but anyway obscures intuition.

Group theory basics

Group action $\langle G, A \rangle$

Group $G = \langle \underline{G}, \ldots \rangle$

G-**action** $A = \langle \underline{A}, \ldots \rangle$

Intuition

- **symmetries**: $\underline{G}$ elements
- x **symmetric to** y **via** g

action and group axioms mean:

'**being symmetric**' is a
proof-relevant equivalence relation tracked by
symmetries

$G{:=} \quad {=:}A$

Board rotations $\langle \mathbf{C}_4, \mathrm{Dir} \rangle$

$\underline{\mathbf{C}_4} := \{\curvearrowleft, \curvearrowright, \curvearrowright, \circlearrowright\}$

acting on

$\underline{\mathrm{Dir}} := \{\Leftarrow, \Rightarrow, \Uparrow, \Downarrow\}$

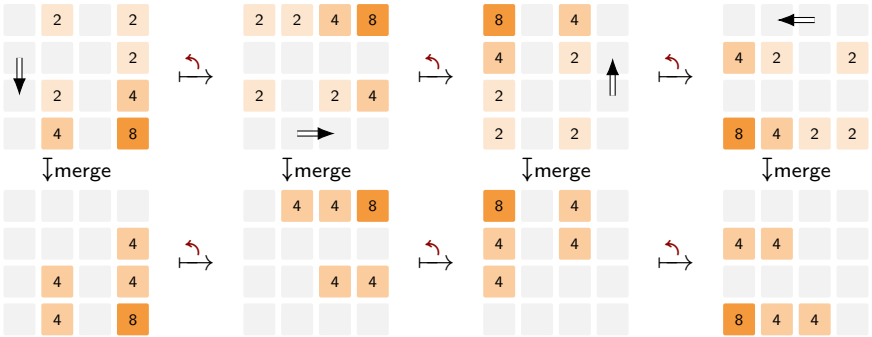
$\Uparrow$ symmetric to $\Leftarrow$ via $\curvearrowleft$
$\Downarrow$ symmetric to $\Leftarrow$ via $\curvearrowright$
$\Rightarrow$ symmetric to $\Leftarrow$ via $\curvearrowright$
$\Leftarrow$ symmetric to $\Leftarrow$ via $\circlearrowright$

Group theory basics

Equivariant map $\varphi : A \circledast\!\!\longrightarrow B$ between G-actions A, B

$$\frac{x, \quad y \text{ symmetric via } g}{f\,x, f\,y \text{ symmetric via } g}$$

Example: merge : Dir $\times$ Board$_n$ $\circledast\!\!\longrightarrow$ Board$_n$



Group theory basics

Group action $\langle G, A \rangle$

Group $G = \left\langle \underline{G}, (*), e, (-)^{-1} \right\rangle$:

$\underline{G}$(**carrier** set) $\quad \dfrac{g, h : \underline{G}}{g * h : \underline{G}}$(**multiplication**/**composition**)

$\dfrac{}{e : \underline{G}}$(**unit**/**identity**/**neutral**) $\quad \dfrac{g : \underline{G}}{g^{-1} : \underline{G}}$(**inverse**)

such that:

$$\mathsf{g}, \mathsf{h}, \mathsf{k} : \underline{G} \vdash (\mathsf{g} * \mathsf{h}) * \mathsf{k} = \mathsf{g} * (\mathsf{h} * \mathsf{k}) \qquad \text{(associativity)}$$

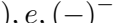
$$\mathsf{g} : \underline{G} \vdash \mathsf{g} * e = \mathsf{g} = e * \mathsf{g} \qquad \text{(neutrality)}$$

$$\mathsf{g} : \underline{G} \vdash \mathsf{g} * \mathsf{g}^{-1} = e = \mathsf{g}^{-1} * \mathsf{g} \qquad \text{(inverses)}$$

G-**action** $A = \langle \underline{A}, \ldots \rangle \ldots$

Board rotations $\langle \mathbf{C}_4, \mathrm{Dir} \rangle$ ($G :=$, $=: A$)

$\underline{\mathbf{C}_4} := \{$  $,$ $,$ $,$ $\}$



group laws imply rest and:

$$e = \text{}$$

Group theory basics

Group action $\langle G, A \rangle$

Group $G = \left\langle \underline{G}, (*), e, (-)^{-1} \right\rangle$

G-**action** $A = \langle \underline{A}, \overset{A}{\circledast} \rangle$:

$\underline{A}$(**carrier** set) $\qquad \dfrac{g : \underline{G} \qquad x : \underline{A}}{g \circledast x : \underline{A}}$(**action** of G on $\underline{A}$)

such that:

$$\mathsf{g}, \mathsf{h} : \underline{G}, \mathsf{x} : \underline{A} \vdash \mathsf{g} \circledast (\mathsf{h} \circledast \mathsf{x}) = (\mathsf{g} * \mathsf{h}) \circledast \mathsf{x} \quad \text{(associativity)}$$

$$\mathsf{x} : \underline{A} \vdash e \circledast \mathsf{x} = \mathsf{x} \quad \text{(neutrality)}$$

Board rotations $\langle \underline{\mathbf{C}_4}, \mathsf{Dir} \rangle$ ($G := \underline{\mathbf{C}_4}$, $A := \mathsf{Dir}$)

$\underline{\mathbf{C}_4} := \{\curvearrowleft, \curvearrowright, \curvearrowright, \circlearrowright\}$

$\curvearrowleft * \curvearrowleft := \curvearrowright \quad \curvearrowleft * \curvearrowright := \curvearrowright$

$\curvearrowleft * \curvearrowright := \circlearrowright \quad \curvearrowleft * \circlearrowright := \curvearrowleft$

acting on

$\underline{\mathsf{Dir}} := \{\Leftarrow, \Rightarrow, \Uparrow, \Downarrow\}$

$\curvearrowleft \circledast \Leftarrow := \Downarrow \quad \curvearrowleft \circledast \Downarrow := \Rightarrow$

$\curvearrowleft \circledast \Rightarrow := \Uparrow \quad \curvearrowleft \circledast \Uparrow := \Leftarrow$

laws determine the rest acting similarly on: Board_n and $\mathsf{Dir} \times \mathsf{Board}_n$

Group theory basics

Group action $\langle G, A \rangle$

Group $G = \left\langle \underline{G}, (*), e, (-)^{-1} \right\rangle$

G-**action** $A = \langle \underline{A}, \rangle$

Notation

- **symmetries**: $\underline{G}$ elements
- x **symmetric to** y **via** g when: $g \circledast x = y$

action and group axioms mean:

'**being symmetric**' is a
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$G{:=} \quad {=:}A$

Board rotations $\langle \mathbf{C}_4, \mathrm{Dir} \rangle$

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laws determine the rest
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Board_n and $\mathrm{Dir} \times \mathrm{Board}_n$

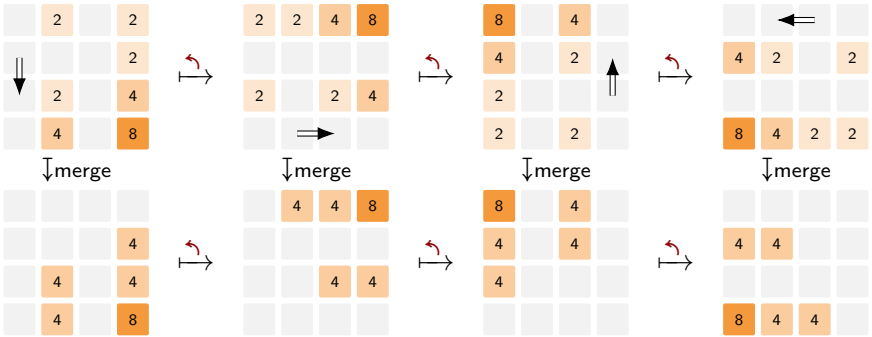
Group theory basics

Equivariant map $\varphi : A \circledast\!\!\longmapsto B$ between G-actions A, B

$$\varphi : \underline{A} \to \underline{B} \quad \text{function} \qquad \mathsf{g} : \underline{G}, \mathsf{x} : \underline{A} \vdash \varphi(\mathsf{g} \circledast \mathsf{x}) = \mathsf{g} \circledast (\varphi \mathsf{x}) : \underline{B}$$

i.e.: $\dfrac{x, \quad y \text{ symmetric via } g}{f\,x, f\,y \text{ symmetric via } g}$

Example: merge : Dir $\times$ Board$_n$ $\circledast\!\!\longmapsto$ Board$_n$



Talk structure

- Introduction
- Group theory basics
- Simply-typed symmetric programming
- Dependently-typed symmetric programming
  - Indexed actions and w.l.o.g.
  - Mathematical reasoning
- Conclusion

Not today / offline

- Dijkstra'95 & Harrison'09 w.l.o.g. principles
- Symmetric user-defined data-types
  - Simply-typed data-types
  - Indexed data-types
- Equivariant reasoning

w.l.o.g. anatomy: canonisation

w.l.o.g. = (input & output symmetries) +(**canonising** symmetries) +(core function/proof)

Canoniser for G-action A
function $c : \underline{A} \to \underline{G}$ assigning **canonising symmetries** to A-elements.

Canonical elements

$$\mathsf{Can}\, c := \{\, (c\, a) \circledast a \mid a \in \underline{A} \,\}$$

Example: mkLft : $\underline{\mathsf{Dir}} \to \underline{\mathbf{C}_4}$

Canonising symmetry makes direction 'left':

$\mathsf{mkLft} \Rightarrow := \curvearrowright$ $\quad \mathsf{mkLft} \Uparrow := \curvearrowleft$

$\mathsf{mkLft} \Leftarrow := \circlearrowleft$ $\quad \mathsf{mkLft} \Downarrow := \curvearrowright$

Only canonical direction is 'left': $\mathsf{Can}\, \mathsf{mkLft} = \{\Leftarrow\}$

w.l.o.g. anatomy: canonisation

Canonical elements

$$\mathsf{Can}\, c \coloneqq \{\, (c\, a) \circledast a \mid a \in \underline{A} \,\}$$

Lemma

$$\mathsf{Can}\left(\underline{A} \times \underline{B} \xrightarrow{\pi_1} \underline{A} \xrightarrow{c} \underline{G}\right) = \mathsf{Can}\left(\underline{A} \xrightarrow{c} \underline{G}\right) \times \underline{B}$$

Example

Canonise merge-inputs by canonising the direction:

$$c : \underline{\mathsf{Dir} \times \mathsf{Board}_n} \xrightarrow{\pi_1} \underline{\mathsf{Dir}} \xrightarrow{\mathsf{mkLft}} \underline{\mathbf{C}_4}$$

By this Lemma, in canonical merge-inputs, merge-direction left but board unrestricted:

w.l.o.g. anatomy: core

w.l.o.g. = (input & output **symmetries**)+(**canonising** symmetries) +(**core** function/proof)

Core function $f : \mathsf{Can}\, c \to \underline{B}$

work on canonical elements, typically more concrete, after eliminating symmetric cases

Example: merging left

$$
\begin{aligned}
&\mathsf{merge}^{\mathrm{row}}_{\Leftarrow} && : \mathsf{Row}_n \to \mathsf{Board}_n \\
&\mathsf{merge}^{\mathrm{row}}_{\Leftarrow}\ \langle\rangle && := \langle\rangle \\
&\mathsf{merge}^{\mathrm{row}}_{\Leftarrow}\ \langle 0\rangle +\!\!+ r && := (\mathsf{merge}^{\mathrm{row}}_{\Leftarrow}\ r) +\!\!+ \langle 0\rangle \\
&\mathsf{merge}^{\mathrm{row}}_{\Leftarrow}\ \langle x\rangle && := \langle x\rangle \\
&\mathsf{merge}^{\mathrm{row}}_{\Leftarrow}\ (\langle x\ , 0\ \rangle +\!\!+ r) && := \mathsf{merge}_{\Leftarrow}\ (\langle x\rangle +\!\!+ r) +\!\!+ \langle 0\rangle \\
&\mathsf{merge}^{\mathrm{row}}_{\Leftarrow}\ \left(\left\langle 2^k, 2^k\right\rangle +\!\!+ r\right) && := \left\langle 2^{k+1}\right\rangle +\!\!+ (\mathsf{merge}^{\mathrm{row}}_{\Leftarrow}\ r) +\!\!+ \langle 0\rangle \\
&\mathsf{merge}^{\mathrm{row}}_{\Leftarrow}\ \left(\left\langle 2^k, 2^\ell\right\rangle +\!\!+ r\right) && := \left\langle 2^k\ \right\rangle +\!\!+ \mathsf{merge}^{\mathrm{row}}_{\Leftarrow}\ \left(\left\langle 2^\ell\right\rangle +\!\!+ r\right)
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{merge}_{\Leftarrow} : \mathsf{Board}_n \to \mathsf{Board}_n \\
&\mathsf{merge}_{\Leftarrow} := \mathsf{map}\ \mathsf{merge}^{\mathrm{row}}_{\Leftarrow}
\end{aligned}
$$

Symmetric programming **abstraction**

w.l.o.g. = (input & output **symmetries**)+(**canonising** symmetries)+(**core** function/proof)

W.l.o.g. construction

Formally:

$$\frac{G\text{-actions } A, B \qquad c : \underline{A} \to \underline{G} \qquad f : \mathsf{Can}\, c \to \underline{B}}{\mathsf{wlog}(c, f) \coloneqq \Big(a \mapsto (ca)^{-1} \circledast f((ca) \circledast a)\Big) : \underline{A} \to \underline{B}}$$

Example

```
mergeUnchecked : {n : Nat} -> Dir -> Board n -> Board n
mergeUnchecked = Prelude.curry $ wlog
        (\(dir, _) => makeLeft dir)                    -- canoniser
        (\case (Lt, board) => mergeLeft board          -- core function
               _ => believe_me "oops")                 --   unreachable case
```

Symmetric programming **abstraction**

w.l.o.g. = (input & output **symmetries**)+(**canonising** symmetries)+(**core** function/proof)

W.l.o.g. construction

Formally:

$$\frac{G\text{-actions } A, B \qquad c : \underline{A} \to \underline{G} \qquad f : \mathsf{Can}\, c \to \underline{B}}{\mathsf{wlog}(c, f) \coloneqq \Big(a \mapsto (ca)^{-1} \circledast f((ca) \circledast a)\Big) : \underline{A} \to \underline{B}}$$

Example

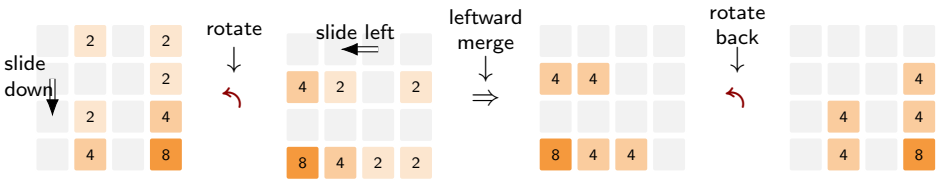
```
core : {n : Nat} -> Canon (Canoniser {n}) -> Board n
core  (( d, b) ** ev) with 0 (canonicalView ev)
 core ((Lt, b) ** ev) | Refl = mergeLeft b

merge : {n : Nat} -> Dir -> Board n -> Board n
merge = Prelude.curry $ wlog' Canoniser core
```

Symmetric programming in Idris 2 [Brady'21]

Simply-typed w.l.o.g.

- Straightforward to implement
- Straightforward to use
- Plumbing group actions with type-classes/interfaces fiddly as usual
- Plenty of 'believe_me' in code for unreachable cases
- Doesn't cover mathematical reasoning



TwentyFortyEight.idr

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  - Simply-typed data-types
  - Indexed data-types
- Equivariant reasoning

Back to mathematics

these concepts apply to **propositions** & **proofs**

concepts apply **uniformly** via dependent types:

types as propositions; programs as proofs

$\wr$

G-**actions** on propositions; **canonisers** and **core functions** in wlog proofs

easy with **simple types** (HOL/FOL), **duplicating** concepts in different **kinds/stages** (terms vs. propositions, object-level vs. meta-level)

analyse extant w.l.o.g. principles of Dijkstra & Harrison

Explicating type dependency

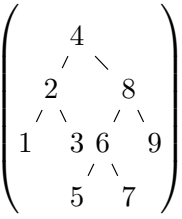
Family $\Gamma \vdash X$

- Γ set of **indices**
- X a family assigning a set X_γ to each index $\gamma \in \Gamma$—the **fibre** over γ

Example: internal tree nodes $\mathsf{t} : \mathsf{Tree}\, S \vdash \mathsf{Node}\, \mathsf{t}$

- index $\mathsf{Tree}\, S \ni t$ binary tree with labels in S;
- fibre $\mathsf{Node}\, t$: paths ending in internal node of $\mathsf{Tree}\, S \ni t$.

$\mathsf{R} :: \mathsf{L} :: \mathsf{Nil} \in \mathsf{Node}$



Fibres collectively form the smallest sets of lists over $\{\mathsf{L}, \mathsf{R}\}$ containing:

$$\frac{}{\mathsf{Nil} \in \mathsf{Node}} \qquad \frac{p \in \mathsf{Node}\, \ell}{(\mathsf{L} :: p) \in \mathsf{Node}} \qquad \frac{p \in \mathsf{Node}\, r}{(\mathsf{R} :: p) \in \mathsf{Node}}$$

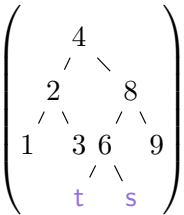
Explicating type dependency

Term/fibred element $\Gamma \vdash M : X$

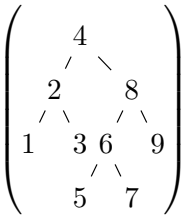
- family $\Gamma \vdash X$
- M a sequence assigning an element $M_\gamma \in X_\gamma$ at every fibre over $\gamma \in \Gamma$

Example

$\mathsf{t}, \mathsf{s} : \mathsf{Tree}\,\mathbb{N} \vdash \mathsf{R} :: \mathsf{L} :: \mathsf{Nil} : \mathsf{Node}$



element at fibre $\mathsf{t} \mapsto 5, \mathsf{s} \mapsto 7$: $\mathsf{R} :: \mathsf{L} :: \mathsf{Nil} \in \mathsf{Node}$



Formalising type dependency

Abstractions for type dependency:

$$\frac{\Gamma \vdash X}{\Gamma, \mathsf{x} : X \text{ set}} \qquad \frac{\theta : \Gamma \to \Delta \qquad \Delta \vdash Y}{\Gamma \vdash Y\,[\theta]} \qquad \frac{\Gamma \vdash X}{\mathsf{wkn} : (\Gamma, \mathsf{x} : X) \to \Gamma} \qquad \frac{\Gamma \vdash X}{\Gamma, \mathsf{x} : X \vdash \mathsf{x} : X\,[\mathsf{wkn}]}$$

$$\frac{\theta : \Delta \to \Gamma \qquad \Delta \vdash M : X\,[\theta]}{(\theta, \mathsf{x} \mapsto M) : \Delta \to (\Gamma, \mathsf{x} : X)} \qquad \frac{\Gamma \vdash X \qquad \Gamma, \mathsf{x} : X \vdash Y}{\Gamma \vdash (\mathsf{x} : X) \times Y} \qquad \frac{\Gamma \vdash M : X \quad \Gamma \vdash K : Y\,[\mathsf{x} \mapsto M]}{\Gamma \vdash M, K : (\mathsf{x} : X) \times Y}$$

$$\frac{\Gamma \vdash M : (\mathsf{x} : X) \times Y}{\Gamma \vdash M.\mathsf{fst} : X} \qquad \frac{\Gamma \vdash M : (\mathsf{x} : X) \times Y}{\Gamma \vdash M.\mathsf{snd} : Y\,[\mathsf{x} \mapsto M.\mathsf{fst}]} \qquad \frac{\Gamma \vdash X \;\; \Gamma, \mathsf{x} : X \vdash Y}{\Gamma \vdash (X : \mathsf{x}) \to Y}$$

$$\frac{\Gamma \vdash K : (X : \mathsf{x}) \to Y \qquad \Gamma \vdash M : X}{\Gamma \vdash K\,M : Y\,[\mathsf{x} \mapsto M]} \qquad \frac{\Gamma, \mathsf{x} : X \vdash M : Y}{\Gamma \vdash (\lambda \mathsf{x} : X.M) : (\mathsf{x} : X) \to Y}$$

Formalising type dependency

Families defer type-dependency difficulties to meta-level:

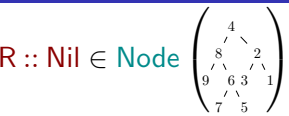
- Dependent-pairs/Grothendieck construction $\coprod_\Gamma X \coloneqq \{\gamma, x \mid \gamma \in \Gamma, x \in X_\gamma\}$
- Dependent-functions/products $\prod_\Gamma X$

Dependently-typed actions

Indexed G-action $\Gamma \vdash A$

- $\Gamma = \langle \underline{\Gamma}, (\circledast) \rangle$ **indexing** G-action
- $\underline{\Gamma} \vdash \underline{A}$ **carrier** family
- $$\frac{\vdash g : \underline{G} \qquad \gamma : \Gamma \vdash a : \underline{A}}{\gamma : \Gamma \vdash g \underset{\gamma}{\circledast} a : \underline{A}\left[\gamma \mapsto g \circledast \gamma\right]}$$ Γ-**indexed action** of G on $\underline{A}$

$$\curvearrowright\!\circledast\,(\mathsf{R} :: \mathsf{L} :: \mathsf{Nil}) = \mathsf{L} :: \mathsf{R} :: \mathsf{Nil} \in \mathsf{Node}$$



such that

$$\gamma : \Gamma, \mathsf{x} : \underline{A} \vdash e \underset{\gamma}{\circledast} \mathsf{x} = \mathsf{x} \qquad \mathsf{g}, \mathsf{k} : \underline{G}, \gamma : \underline{\Gamma}, \mathsf{x} : \underline{A} \vdash (\mathsf{g} * \mathsf{k}) \underset{\gamma}{\circledast} \mathsf{x} = \mathsf{g} \underset{\mathsf{k} \circledast \gamma}{\circledast} (\mathsf{k} \underset{\gamma}{\circledast} \mathsf{x})$$

equivalently: $\quad (\circledast) : (\mathsf{g} : \underline{G}) \to (\gamma : \Gamma) \to \underline{A}_{\gamma} \to \underline{A}_{\mathsf{g} \circledast \gamma}$

Proposition

Indexed actions amount to equivariant projections from the Grothendieck construction:

$$\underset{\text{indexed } G\text{-action}}{\Gamma \vdash A} \iff \underset{G\text{-action}}{\coprod_{\Gamma} A = (\coprod_{\underline{\Gamma}} \underline{A}, ((\circledast), (\circledast)))} \qquad \pi_1 : \coprod_{\Gamma} A \circledast\!\!\to \Gamma$$

Indexed w.l.o.g. anatomy: canonisers

Indexed canoniser for G-action $\Gamma \vdash A$

term $\underline{\Gamma} \vdash c : \underline{A} \to \underline{G}$

induced **family** $\underline{\Gamma} \vdash \mathsf{Can}\, c$ of canonical elements:

$$(\mathsf{Can}\, c)_\gamma := \left\{ \left. {}_{g^{-1} \circledast \gamma}(g \circledast a) \,\right|\, g \in \underline{G}, a \in \underline{A}_{g^{-1} \circledast \gamma}, c_{g^{-1} \circledast \gamma} a = g \right\}$$

Example

A canoniser for paths indexed by binary trees with action of $\mathbf{S}_2 := \{\mathtt{I}, \curvearrowright\}$:

$\mathsf{t} : \mathsf{Tree}\, \mathbb{N} \vdash c : \mathsf{Node}\, \mathsf{t} \to \mathbf{S}_2$

$$\begin{aligned} c_{\mathsf{Node}\langle \ell, n, r\rangle}\, \mathsf{Nil} &:= \mathtt{I} \\ c_{\mathsf{Node}\langle \ell, n, r\rangle}(\mathsf{L} :: p) &:= \mathtt{I} \\ c_{\mathsf{Node}\langle \ell, n, r\rangle}(\mathsf{R} :: p) &:= \curvearrowright \end{aligned}$$

Canonical paths may only turn R after first step:

$$(\mathsf{Can}\, c)_{\mathsf{Node}\langle \ell, n, r\rangle} = \{\mathsf{Nil}\} \cup \{\mathsf{L} :: p \mid p \in \mathsf{Node}\, \ell\}$$

Indexed w.l.o.g. anatomy: core functions

Indexed core function

term $\underline{\Gamma} \vdash f : (\mathsf{x} : \mathsf{Can}\, c) \to \underline{B}_{\mathsf{x}}$

Example

core function returning node pointed at by a path:

$$\begin{aligned} \mathsf{t} : \mathsf{Tree}\,\mathbb{N} \vdash \mathsf{lookup}^{\mathrm{Core}} &: \mathsf{Can}\, c \to \mathbb{N} \\ \mathsf{lookup}^{\mathrm{Core}}_{\mathsf{Node}\langle \ell, \mathsf{m}, r\rangle}\, \mathsf{Nil} &\coloneqq \mathsf{m} \\ \mathsf{lookup}^{\mathrm{Core}}_{\mathsf{Node}\langle \ell, \mathsf{m}, r\rangle}\, (\mathsf{L} :: p) &\coloneqq \mathsf{lookup}_{\ell} p \end{aligned}$$

✓ **eliminated** symmetric case

✗ **recursive** call to lookup, not $\mathsf{lookup}^{\mathrm{Core}}$
revisit through **data-types** (not today)

Symmetric programming **dependently-typed** abstraction

indexed w.l.o.g. = (**indexed** symmetries)+(**indexed** canoniser)+(**indexed** core function)

W.l.o.g. construction

Formally:

$$\frac{G\text{-actions } \Gamma \vdash A \qquad \Gamma, \mathsf{x} : A \vdash B \qquad \Gamma \vdash c : \underline{A} \to \underline{G} \qquad \Gamma \vdash f : \mathsf{Can}\, c \to \underline{B}}{\gamma : \underline{\Gamma} \vdash \mathsf{wlog}(c, f) \coloneqq \left(\lambda a.(ca)^{-1} \underset{(ca) \circledast \gamma}{\circledast} f((ca) \underset{\gamma}{\circledast} a) \right) : (\mathsf{x} : \underline{A}) \to \underline{B}}$$

Example (revisit later)

$$\mathsf{t} : \mathsf{Tree}\, \mathbb{N} \vdash \mathsf{lookup} \coloneqq \mathsf{wlog}(c, \mathsf{lookup}^{\mathrm{Core}}) : \mathsf{Node}\, \mathsf{t} \to \mathbb{N}$$

Talk structure

- Introduction
- Group theory basics
- Simply-typed symmetric programming
- Dependently-typed symmetric programming
  - Indexed actions and w.l.o.g.
  - Mathematical reasoning
- Conclusion

Not today / offline

- Dijkstra'95 & Harrison'09 w.l.o.g. principles
- Symmetric user-defined data-types
  - Simply-typed data-types
  - Indexed data-types
- Equivariant reasoning

Mathematical reasoning w.l.o.g.

Propositions as types, proofs as programs

For this part, take propositions $\varphi \in \mathsf{Prop} := \{\mathsf{True}, \mathsf{False}\}$. Use trivial action as needed.

Prop acts as a universe with:

$$\frac{\Gamma \vdash \varphi : \mathsf{Prop}}{\Gamma \vdash \varphi} \qquad \varphi_\gamma := \begin{cases} \varphi_\gamma = \mathsf{True} : & \mathbb{1} \\ \text{otherwise} : & \emptyset \end{cases}$$

E.g.: $(\mathsf{x}, \mathsf{y}, \mathsf{z}) : (0, \infty)^3 \vdash \mathsf{Schur} :=$

$$\forall t : \mathbb{R}. x_1^t(x_1 - x_2)(x_1 - x_3) + x_2^t(x_2 - x_3)(x_2 - x_1) + x_3^t(x_3 - x_1)(x_3 - x_2) \geq 0 : \mathsf{Prop}$$

At most one term $\Gamma \vdash M : \varphi$. It exists iff φ holds in every fibre $\gamma \in \Gamma$.

Equivariant terms $\Gamma \vdash M : A$ (where $\Gamma \vdash A$ indexed G-action)

terms $\underline{\Gamma} \vdash M : \underline{M}$ satisfying: $\mathsf{g} : \underline{G}, \gamma : \underline{\Gamma} \vdash g \underset{\gamma}{\circledast} M_\gamma = M_{\mathsf{g} \circledast \gamma} : A_{\mathsf{g} \circledast \gamma}$

Proposition

$\Gamma \vdash \varphi : \mathsf{Prop}$ **invariant** *i.e., equivariant w.r.t. trivial action,* $\iff$ $\Gamma \vdash \varphi$ *has an indexed action (this indexed action is unique)*

Mathematical reasoning w.l.o.g.

Revisionism

Few sources treat w.l.o.g. formally, we don't encompass all extant uses

Use our w.l.o.g. principle for **taxonomy**:

- ✓ Dijkstra'95 [EWD1223, not mechanised]
- ✓ HOL WLOG tactics [Harrison'09]
- ✗ Rocq SSReflect
- ✓ Agda Stdlib
- ✗ Lean

Road-map

More in the draft

- Analysis of Dijkstra's and Harrison's w.l.o.g. principles
- Extension to indexed data-types
- More examples:
  - lookup without Maybe;
  - insertion to McBride's binary search trees
- Meta-theoretic results for equivariant reasoning:
  - representation theorem for equivariant maps
  - completeness of w.l.o.g. reasoning

Future work

- more examples, connections nominal techniques and symmetry breaking
- mechanisation
- efficient implementation

Contribution

Symmetric programming

Exploit symmetries for computation and reasoning—**avoid symmetric cases**.
Mathematical foundations for symmetric programming semantics:

- **Abstractions** for symmetric programming
- **Simply-typed** reasoning about equivariance
- **Dependently-typed** and mathematical reasoning
- User-defined **data-types** through initial algebra semantics and induction principles
  - Simply-typed data-types
  - Dependently-typed/indexed data-types