Denotational validation of higher-order Bayesian inference

Yufei Cai, Zoubin Ghahramani, Chris Heunen, Ohad Kammar, Sean K. Moss, Klaus Ostermann, Adam Ścibior, Sam Staton, Matthijs Vákár, and Hongseok Yang

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What is probabilistic programming?

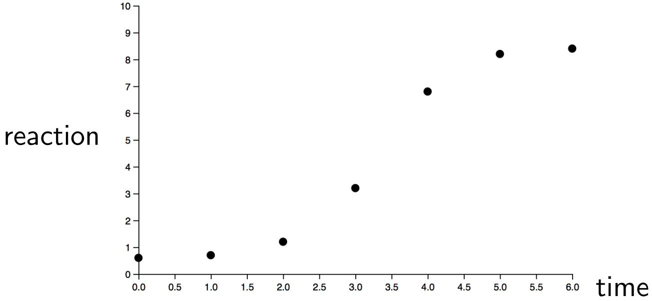
Bayesian data modelling

1. Develop a probabilistic (generative) model.
2. Design an inference algorithm for the model.
3. Using the algorithm, fit the model to the data.

What is probabilistic programming?

Example

Effect of a drug on a patient, given data:



What is probabilistic programming?

Generative model

$$
\begin{aligned}
s &\sim \mathsf{normal}(0, 2) \\
b &\sim \mathsf{normal}(0, 6) \\
f(x) &= s \cdot x + b \\
y_i &= \mathsf{normal}(f(i), 0.5) \\
&\qquad \text{for } i = 0 \ldots 6
\end{aligned}
$$

What is probabilistic programming?

Generative model

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Conditioning

$y_0 = 0.6, y_1 = 0.7, y_2 = 1.2, y_3 = 3.2, y_4 = 6.8, y_5 = 8.2, y_6 = 8.4$

Predict f?

What is probabilistic programming?

Bayesian inference

$$P(s, b | y_0, \ldots, y_6) = \frac{P(y_0, \ldots, y_6 | s, b) \cdot P(s, b)}{P(y_0, \ldots, y_6)}$$

What is probabilistic programming?

Bayesian inference

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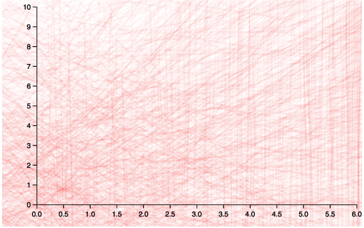

Prior

What is probabilistic programming?

Bayesian inference

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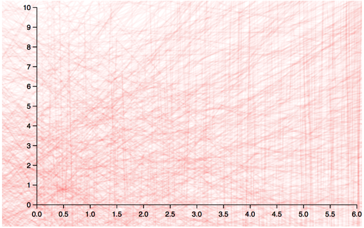
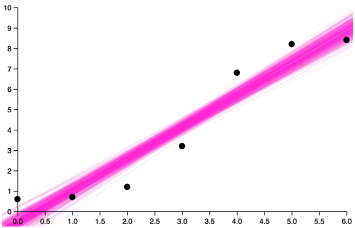


Prior



Posterior

What is probabilistic programming?

Probabilistic programming models

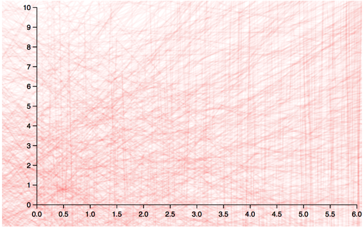
1. Develop a probabilistic (generative) model.
   Write a program.
2. ~~Design an inference algorithm for the model.~~
3. Using the built-in algorithm, fit the model to the data.

What is probabilistic programming?

In Anglican [Wood et al.'14]

```
(let [s (sample (normal 0.0 2.0))
      b (sample (normal 0.0 6.0))
      f (fn [x] (+ (* s x) b)))]
```

```
(predict :f f))
```



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(let [s (sample (normal 0.0 2.0))
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  (observe (normal (f 1.0) 0.5) 2.5)
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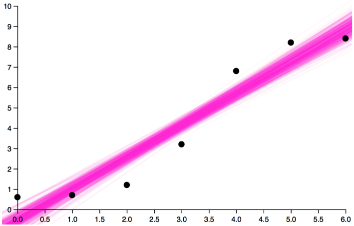

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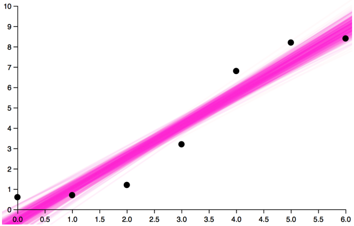

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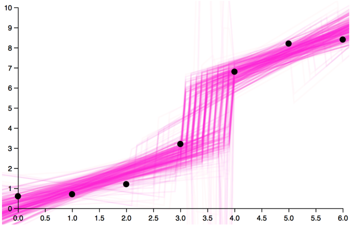

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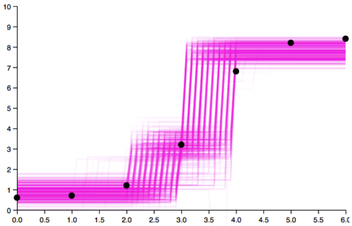

What is probabilistic programming?

Components

- Control flow, e.g.: simply typed λ-calculus
- data types, e.g.: lists, functions, thunks
- Probabilistic choice: `(sample (normal 0.0 2.0))`
- Conditioning: `(observe (normal (f 2.0) 0.5) 3.8)`

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$$\text{posterior} \propto \text{liklihood} \times \text{prior}$$

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$$\text{posterior} \propto \text{liklihood} \times \text{prior}$$

Which we refine to:

$$\text{posterior} = \text{weight} \odot \text{prior}$$

Some measure theory

Rescaling

$$\nu = w \odot \mu$$

when for all $\chi : X \to [0, \infty]$:

$$\int_X \chi(x)\nu(\mathrm{d}x) = \int_X \chi(x) \cdot w(x)\mu(\mathrm{d}x)$$

(where X measurable space, $\mu \in MX$ measures on X,
$w : X \to [0, \infty]$ measurable function)

Theorem (Radon-Nikodym)

For all finite ν, μ: if such w exists, then it is unique μ-almost everywhere.

Write: $\nu \ll \mu$, $w = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$

What is probabilistic programming?

A probabilistic program is a measure

For $t : X$

$$[\![t]\!] = w \odot \text{prior}\,[\![t]\!]$$

where prior $[\![t]\!]$ is the **prior** (ignore conditioning),
and $w = \frac{\mathrm{d}[\![t]\!]}{\mathrm{d}(\text{prior}[\![t]\!])}$

Conditioning

$$\frac{t : x \qquad \varphi : X \to [0, +\infty]}{\texttt{observe}(t, \varphi) : 1}$$

and

$$[\![\texttt{observe}]\!]\,(x, \varphi) = \varphi(x) \odot \delta_{()}$$

What is probabilistic programming?

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Conditioning

Replace `observe` by score :

$$\frac{r : [0, \infty]}{\mathsf{score}\, r : 1}$$

and

$$[\![\mathsf{score}\,]\!]\,(r) = r \odot \delta_{()}$$

What is probabilistic programming?

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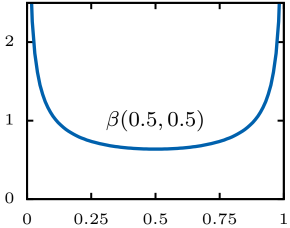
Note

For probability measures prior $[\![t]\!]$:

- It's possible that $\max w > 1$, e.g.:



  or even $\max w = \infty$

- If we insist that all measures are sub-probability measures, then w and $[\![t]\!]$ are **not** compositional (i.e., global)

What is probabilistic programming?

A probabilistic program is an s-finite measure [Staton'17]

For $t : X$

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where prior $[\![t]\!]$ is the **prior** (ignore conditioning),
and $w = \frac{\mathrm{d}[\![t]\!]}{\mathrm{d}(\mathsf{prior}[\![t]\!])}$
Sampling manipulates prior.
Conditioning affects w, sequenced multiplicatively.

S-finite measures

$$\sum_{i \in \mathbb{N}} \mu_i$$

μ_i finite: $\mu_i(X) < \infty$

What is inference?

Computing distributions

For $t : X$

$$[\![t]\!] = w \odot \text{prior}\,[\![t]\!]$$

we want to:

- Plot $[\![t]\!]$.
- Sample $[\![t]\!]$ (e.g., to make prediction)

Challenge

Given a fair coin $(\frac{1}{2}\delta_1 + \frac{1}{2}\delta_0)$, how do we sample from a biased coin $(p\delta_1 + (1-p)\delta_0)$?
Generalise:
Given a prior distribution prior $[\![t]\!]$, how do we sample from $[\![t]\!]$?

What is inference?

Programming-language experts needed

In the traditional areas:

- Verification
- Correctness
- Static analysis
- Semantics
- Optimisation
- Programming abstractions
- Type systems

This talk

Correctness of inference

Inference algorithm: distribution/meaning preserving transformation from one inference representation to another

Requirements

- Represented data is continuous
- Compositional inference representations (IRs)
- IRs are **higher-order**

Traditional measure theory. . .

This talk

Correctness of inference

Inference algorithm: distribution/meaning preserving transformation from one inference representation to another

Requirements

- Represented data is continuous
- Compositional inference representations (IRs)
- IRs are **higher-order**

Traditional measure theory... is unsuitable:

Theorem (Aumann'61)

The set $\mathbf{Meas}(\mathbb{R}, \mathbb{R})$ *cannot be made into a measurable space with*

$$eval : \mathbf{Meas}(\mathbb{R}, \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$$

measurable.

Contribution

Correctness of inference

- Modular validation of inference algorithms:
  Sequential Monte Carlo, Trace Markov Chain Monte Carlo
  By combining:
- Synthetic measure theory [Kock'12]: measure theory without measurable spaces
- Quasi-Borel spaces: a convenient category for higher-order measure theory [LICS'17]

Talk structure

- Probabilistic programming and Bayesian inference
- Synthetic measure theory
- Quasi-Borel spaces
- Inference representations
- Trace Markov Chain Monte Carlo (Trace MCMC)
- Conclusion

Synthetic measure theory: axioms

Measure category [Kock'12]

A pair $(\mathcal{C}, \underline{\mathrm{M}})$

- Cartesian-closed category $\mathcal{C}$

Synthetic measure theory: axioms

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Synthetic measure theory: axioms

Measure category [Kock'12]

A pair $(\mathcal{C}, \underline{\mathrm{M}})$

- Cartesian-closed category $\mathcal{C}$
- Countable coproducts and countable limits
- $\underline{\mathrm{M}} = (\mathrm{M}, \mathrm{return}\,, \gg\!=)$ a strong commutative monad, i.e.:

$$\mathrm{M} : |\mathcal{C}| \to |\mathcal{C}| \qquad \mathrm{return}_X : X \to \mathrm{M}\,X$$

$$\gg\!=_{X,Y} : \mathrm{M}\,X \times (\mathrm{M}\,Y)^X \to \mathrm{M}\,Y$$

satisfying the monad laws and

$$\underline{T}.\mathbf{do}\,\{x \leftarrow a; y \leftarrow b; \mathbf{return}(x, y)\}$$
$$=$$
$$\underline{T}.\mathbf{do}\,\{y \leftarrow b; x \leftarrow a; \mathbf{return}(x, y)\}$$

Synthetic measure theory: axioms

Measure category [Kock'12]

A pair $(\mathcal{C}, \underline{\mathrm{M}})$

- Cartesian-closed category $\mathcal{C}$
- Countable coproducts and countable limits
- $\underline{\mathrm{M}} = (\mathrm{M}, \mathrm{return}, \gg\!=)$ a strong commutative monad, i.e.:
- Canonical morphisms are invertible:

$$\mathrm{M}\,\mathbb{0} \cong \mathbb{1} \qquad \mathrm{M}(\coprod_{n\in\mathbb{N}} X) \cong \prod_{n\in\mathbb{N}} \mathrm{M}\,X$$

Synthetic measure theory: consequences

Surprisingly rich structure

- $0 : \mathbb{1} \to \mathrm{M}\,\mathbb{0}$
- $\sum_{n\in\mathbb{N}} X : \prod_{i\in\mathbb{N}} \cong \mathrm{M}(\coprod_{i\in\mathbb{N}} X) \xrightarrow{\mathrm{M}\,\nabla} \mathrm{M}\,X$
- $R := \mathrm{M}\,\mathbb{1}$ a σ-semiring:

$$(\cdot) : R \times R \xrightarrow{\text{double strength}} R \qquad 1 := \mathrm{return}() \in R$$

- Every algebra is an R-module:

$$\odot : R \times \mathrm{M}\,X \xrightarrow{\text{strength}} \mathrm{M}\,X$$

- Associated affine monad:

$$\mathrm{P}\,X \overset{\mathrm{sub}_X}{\dashrightarrow} \mathrm{M}\,X \underset{\underline{1}}{\overset{\mathrm{M}!}{\rightrightarrows}} R$$

Synthetic measure theory: notation

Kock integration

$$\oint_X f(x)\underline{\mu}(\mathrm{d}x) \coloneqq \underline{\mu} \gg= f$$

- Measure-valued, hence analogous to

$$\int_X \chi(x) \cdot f(x)\underline{\mu}(\mathrm{d}x)$$

  for generic $\chi : X \to [0, \infty)$
- η-expanded integrand

Synthetic measure theory: notation

Notation	Meaning	Terminology
R	$:= \mathrm{M}\,\mathbb{1}$	Scalars
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$\underline{\mu}\otimes\underline{\nu}$	$:= \oint_X \left(\oint_Y \underline{\delta}_{(x,y)}\underline{\nu}(\mathrm{d}y)\right)\underline{\mu}(\mathrm{d}x)$	Product measure

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$\mathbb{E}^A_{x\sim\underline{\mu}}[f(x)]$	$:= \underline{\mu} \gg\!= f$	Expectation
$\int_X f(x)\underline{\mu}(\mathrm{d}x)$	$:= \mathbb{E}^R_{x\sim\underline{\mu}}[f(x)]$	Lebesgue integral

Synthetic measure theory: Radon-Nikodym

Radon-Nikodym derivatives

- $\underline{\nu} \ll \underline{\mu}$ when $\underline{\nu} = w \odot \underline{\mu}$;
- w and v are **equal** $\underline{\mu}$**-almost everywhere** when $w \odot \underline{\mu} = v \odot \underline{\mu}$.
- Measurable property: $P : X \to$ bool, induces $[P] : X \to [0, \infty]$
- P over X **holds** $\underline{\mu}$**-a.e.** when $[P] = 1$ $\underline{\mu}$-a.e..

Theorem (Radon-Nikodym)

Let $(\mathcal{C}, \mathrm{M})$ *be a well-pointed measure category. For every* $\underline{\nu} \ll \underline{\mu}$ *in* $\mathrm{M}\,X$*, there exists a* $\underline{\mu}$*-a.e. unique morphism* $\frac{\mathrm{d}\underline{\nu}}{\mathrm{d}\underline{\mu}} : X \to R$ *satisfying* $\frac{\mathrm{d}\underline{\nu}}{\mathrm{d}\underline{\mu}} \odot \underline{\mu} = \underline{\nu}$.

Talk structure

- Probabilistic programming and Bayesian inference
- Synthetic measure theory
- Quasi-Borel spaces
- Inference representations
- Trace Markov Chain Monte Carlo (Trace MCMC)
- Conclusion

Brief measure theory

Measures subsets of $\mathbb{R}$

Borel subsets $\mathcal{B}(\mathbb{R})$ as closure under:

- Intervals $[a, b]$.
- Countable unions.
- Complements.

$\varphi : \mathbb{R} \to \mathbb{R}$ is **measurable** when:

$$B \in \mathcal{B}(\mathbb{R}) \qquad \Longrightarrow \qquad \varphi^{-1}[B] \in \mathcal{B}(\mathbb{R})$$

Source of randomness

Key idea

Propagating randomness from discrete and continuous sampling:

$$\alpha : \mathbb{I} \to X$$

along "random elements":

- for measurable spaces: **derived** through measurable functions;
- for quasi-Borel spaces: **axiomised** through structure.

The category $\mathbf{Qbs}$

Objects

A **quasi-Borel space** $X = \left(|X|, X^{\mathbb{I}}\right)$ consists of:

- a **carrier set** X;
- a set of **random elements** $X^{\mathbb{I}} \subseteq |X|^{\mathbb{I}}$

such that the random elements are closed under:

The category $\mathbf{Qbs}$

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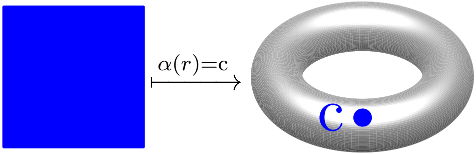
such that the random elements are closed under:

- constant functions $\underline{c}$;



The category Qbs

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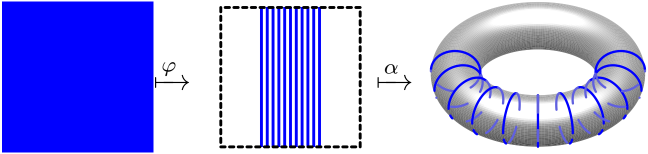

The category $\mathbf{Qbs}$

Objects

A **quasi-Borel space** $X = \left(|X|, X^{\mathbb{I}}\right)$ consists of:

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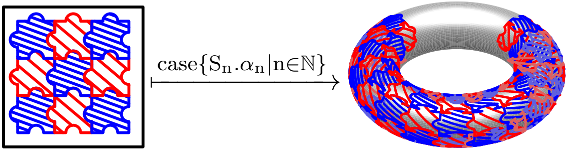
- constant functions $\underline{c}$;
- precomposition with a measurable $\varphi : \mathbb{I} \to \mathbb{I}$
- countable measurable case split.



The category $\mathbf{Qbs}$

Morphisms $f : X \to Y$

Functions $f : |X| \to |Y|$ such that:

$$\alpha \in X^{\mathbb{R}} \qquad \Longrightarrow \qquad f \circ \alpha \in Y^{\mathbb{R}}$$

Subspaces

Every subset $S \subseteq |X|$ inherits the subspace structure:

$$S^{\mathbb{R}} := \left\{ \alpha : \mathbb{R} \to S \middle| \alpha \in X^{\mathbb{I}} \right\}$$

The commutative monad

Measures

(Ω, α, μ):

- Ω is a standard Borel space
- $\alpha \in X^{\Omega}$
- and μ is a σ-finite measure on Ω

Induced integration operator

For $f : X \to [0, \infty]$:

$$\int f \, \mathrm{d}(\Omega, \alpha, \mu) \coloneqq \int_{\Omega} f(\alpha(x)) \, \mu(\mathrm{d}x)$$

Monad of measures

$(\Omega, \alpha, \mu) \approx (\Omega', \alpha', \mu')$ when they determine the same integration operator.

$\mathrm{M}\,X$ consists of equivalence classes of $\approx$.

A synthetic model

The measure category $(\mathbf{Qbs}, \underline{\mathrm{M}})$

- $\mathbf{Qbs}(\mathbb{1}, R) \cong_\sigma [0, \infty]$;
- $\mathbf{Qbs}(R, \mathbb{1} + \mathbb{1}) \cong \mathcal{B}([0, \infty])$ as characteristic functions
- $\mathbf{Qbs}(R, R) \cong \mathbf{Meas}([0, \infty], [0, \infty])$
- Giry $[0, \infty] \rightarrowtail \mathbf{Qbs}(\mathbb{1}, \mathrm{M}(R)) \rightarrowtail$ Measures $[0, \infty]$
- $R^R \times \mathrm{M}(R) \to R$, $(f, \underline{\mu}) \mapsto \int f(x)\, \underline{\mu}(\mathrm{d}x)$ is the Lebesgue integral

Talk structure

- Probabilistic programming and Bayesian inference
- Synthetic measure theory
- Quasi-Borel spaces
- Inference representations
- Trace Markov Chain Monte Carlo (Trace MCMC)
- Conclusion

Representations

Program representation

A **representation** $\underline{T}$ $(T, \mathrm{return}^{\underline{T}}, \gg\!=^{\underline{T}}, m^{\underline{T}})$ consists of:

- $(T, \mathrm{return}^{\underline{T}}, \gg\!=^{\underline{T}})$: monadic interface;
- $m_X^{\underline{T}} : T\,X \to \mathrm{M}\,X$: meaning morphism for every space X

and $m^{\underline{T}}$ preserves $\mathrm{return}^{\underline{T}}$ and $\gg\!=^{\underline{T}}$:

$$\mathrm{return}^{\underline{\mathrm{M}}}\,x = m(\mathrm{return}^{\underline{T}}\,x)$$

$$m(a \gg\!=^{\underline{T}} f) = (m\,a) \gg\!=^{\underline{\mathrm{M}}} \lambda x.\ m(f\ x)$$

Representations

Example representation: lists

$$
\begin{array}{l}
\textbf{instance } \mathit{Rep}\,(\mathsf{List})\ \textbf{where} \\
\quad \textbf{return}\, x \qquad\qquad = [x] \\
\quad x_s \gg\!= f \qquad\qquad = \mathsf{foldr}\,[\ ] \\
\qquad\qquad\qquad\qquad\qquad (\lambda(x, y_s). \\
\qquad\qquad\qquad\qquad\qquad f(x) +\!\!+\, y_s)\ x_s \\
\quad m_{\mathsf{List}}[x_1, \ldots, x_n] = \sum_{i=1}^{n} \underline{\delta}_{x_i}
\end{array}
$$

Representations

Sampling representation

$(T, \mathrm{return}^{\underline{T}}, \gg\!=^{\underline{T}}, m^{\underline{T}}, \mathbf{sample}^{\underline{T}})$

- $(T, \mathrm{return}^{\underline{T}}, \gg\!=^{\underline{T}}, m^{\underline{T}})$: program representation
- $\mathbf{sample}^{\underline{T}} : \mathbb{1} \to T\,\mathbb{I}$

and $m^{\underline{T}} \circ \mathbf{sample}^{\underline{T}} = \mathbf{U}_{\mathbb{I}}$

Representations

Example: free sampler

$\mathsf{Sam}\,\alpha \coloneqq \{\mathsf{Return}\,\alpha \mid \mathsf{Sample}\,(\mathbb{I} \to \mathsf{Sam}\,\alpha)\}$:

$$
\begin{aligned}
&\textbf{instance}\ \textit{Sampling Rep}\,(\mathsf{Sam})\ \textbf{where}\\
&\quad \textbf{return}\,x = \mathsf{Return}\,x\\
&\quad a \gg\!= f \quad = \textbf{match}\,a\,\textbf{with}\,\{\\
&\qquad\qquad\qquad\qquad \mathsf{Return}\,x \to f(x)\\
&\qquad\qquad\qquad\qquad \mathsf{Sample}\,k \to\\
&\qquad\qquad\qquad\qquad\quad \mathsf{Sample}\,(\lambda r.\,k(r) \gg\!= f)\}\\
&\quad \mathrm{sample} \quad = \mathsf{Sample}\,\lambda r.\,(\mathsf{Return}\,r)\\
&\quad m\,a \quad = \textbf{match}\,a\,\textbf{with}\,\{\\
&\qquad\qquad\qquad\qquad \mathsf{Return}\,x \to \underline{\delta}_x\\
&\qquad\qquad\qquad\qquad \mathsf{Sample}\,k \to \oint_{\mathbb{I}} m(k(x))\mathbf{U}(\mathrm{d}x)\}
\end{aligned}
$$

Representations

Conditioning representation

$(T, \mathrm{return}^{\underline{T}}, \gg\!=^{\underline{T}}, m^{\underline{T}}, \mathrm{score}^{\underline{T}})$

- $(T, \mathrm{return}^{\underline{T}}, \gg\!=^{\underline{T}}, m^{\underline{T}})$: program representation
- $\mathrm{score}^{\underline{T}} : [0, \infty) \to T\,\mathbb{1}$

and $m^{\underline{T}} \circ \mathrm{score}^{\underline{T}}\, r = r \odot \underline{\delta}_{()}$

Representations

Weighted values

For every representation $\underline{T}$, $\mathsf{W}\,\underline{T}\,X := T(\mathbb{R}_+ * X)$

$$
\begin{aligned}
&\textbf{instance } \mathit{Conditioning\ Rep}\,(\mathsf{W}\,\underline{T})\ \textbf{where}\\
&\quad \textbf{return}_{\mathsf{W}\,\underline{T}}\, x = \mathrm{return}^{\underline{T}}(1, x)\\
&\quad a \gg\!=_{\mathsf{W}\,\underline{T}} f \quad = \underline{T}.\textbf{do}\,\{(r, x) \leftarrow a;\\
&\qquad\qquad\qquad\qquad (s, y) \leftarrow f(x);\\
&\qquad\qquad\qquad\qquad \textbf{return}(r \cdot s, y)\}\\
&\quad m_{\mathsf{W}\,\underline{T}}\, a \quad = \lambda x.\ \oint_{\mathbb{R}_+ \times X} r \odot \underline{\delta}_x m^{\underline{T}}(a)(\mathrm{d}r, \mathrm{d}x)\\
&\quad \textbf{score}_{\mathsf{W}\,\underline{T}}\, r \quad = \mathrm{return}^{\underline{T}}(r, ())
\end{aligned}
$$

Representations

Inference representation

$(T, \mathrm{return}^{\underline{T}}, \gg\!=^{\underline{T}}, \mathbf{sample}^{\underline{T}}\, \mathrm{score}^{\underline{T}}, m^{\underline{T}})$: sampling and conditioning

Example: weighted sampler

$\mathsf{WSam}\, X := \mathsf{W}\, \mathsf{Sam}\, X = \mathsf{Sam}([0, \infty) \times X)$

Inference transformations

$\underline{t} : \underline{T} \to \underline{S}$
$\underline{t} : T\,X \to S\,X$ for every space X such that:

$$m_{\underline{S}} \circ \underline{t} = m_{\underline{T}}$$

A single compositional step in an inference algorithm

Inference transformations

$\underline{t} : \underline{T} \to \underline{S}$

$\underline{t} : T\,X \to S\,X$ for every space X such that:

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A single compositional step in an inference algorithm

Unnaturality

$\mathrm{aggr}_X : \mathsf{List}(\mathbb{R}_+ * X) \to \mathsf{List}(\mathbb{R}_+ * X)$

aggregating (r, x), (s, x) to $(r + s, x)$

Then $\mathrm{aggr} : \underline{\mathsf{List}} \to \underline{\mathsf{List}}$ but not natural:

$$\mathrm{aggr} \circ \mathsf{List}!\ [(\tfrac{1}{2}, \mathsf{False}), (\tfrac{1}{2}, \mathsf{True})][(1, ())] \\ \neq [(\tfrac{1}{2}, ()), (\tfrac{1}{2}, ())]\ \mathsf{Enum}! \circ \mathrm{aggr}\ [(\tfrac{1}{2}, \mathsf{False}), (\tfrac{1}{2}, \mathsf{True})]$$

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Markov Chain Monte Carlo

Metropolis-Hastings update

$$\begin{aligned}\underline{T}.\mathbf{do}\,\{&x \leftarrow a;\\ &y \leftarrow \psi_a(x);\\ &r \leftarrow \text{sample};\\ &\mathbf{if}\ r < \min(1, \rho_a(x,y))\\ &\mathbf{then\ return}\, y\\ &\mathbf{else\ return}\, x\}\end{aligned}$$

where $\psi_a : X \to T\,X$ and $\rho_a : X \times X \to \overline{\mathbb{R}}_+$

Markov Chain Monte Carlo: abstract foundation

Theorem (Metropolis-Hastings-Green for quasi-Borel spaces)

Given X, $a \in \mathrm{M}\,X$, $\psi_a : X \to \mathrm{M}\,X$, *and* $\rho_a : X \times X \to \overline{\mathbb{R}}_+$, *set* $\underline{\mu}_a \coloneqq [\rho \neq 0] \odot (\iiint \underline{\delta}_{(x,y)} a(\mathrm{d}x)\psi(x, \mathrm{d}y))$.
Assume that:

1. ψ_a *is Markov:* $\psi(x, X) = 1$;
2. $[1 = (\rho \circ \mathsf{swap}) \cdot \rho]$ *holds* $\underline{\mu}_a$*-a.e.;*
3. $\rho = \frac{\mathrm{d}(\mathsf{swap}_* \underline{\mu}_a)}{\mathrm{d}\underline{\mu}_a}$;
4. $\rho(x, y) = 0 \iff \rho(y, x) = 0$ *for all* $x, y \in X$.

Then $(\eta_{\psi_a,\rho_a})(a) = a$.

Proof mimicks measure theoretic proof, e.g. [Geyer'11]

Trace Markov Chain Monte Carlo: Representation

Program traces

- $t \in \mathsf{WSam}\, X$: program structure representation

Trace Markov Chain Monte Carlo: Representation

Program traces

- $t \in \mathsf{WSam}\,X$: program structure representation
- $p : \mathsf{List}\,\mathbb{I}$ a trace in program t

$$
\begin{aligned}
p \in t = \mathbf{match}\,(p, t)\,\mathbf{with}\,\{ \\
&([\,] \quad , \mathsf{Return}\,x \quad) \to \mathsf{True} \\
&(r :: r_s, \mathsf{Sample}\,f \quad) \to [r_s \in f(r)] \\
&-- \quad \text{any other case:} \\
&(\_ \quad , \_ \quad) \to \mathsf{False}\}
\end{aligned}
$$

Trace Markov Chain Monte Carlo: Representation

Program traces

- $t \in \mathsf{WSam}\, X$: program structure representation
- $p : \mathsf{List}\, \mathbb{I}$ a trace in program t
- $$\sum_{t \in \mathsf{WSam}\, X} \mathrm{Paths}\, t := \left\{(t, p) \in \mathsf{WSam}\, X \times \mathsf{List}\, \mathbb{I} \middle| p \in t\right\}$$
  $$\subseteq \mathsf{WSam}\, X \times \mathsf{List}\, \mathbb{I}$$

Trace Markov Chain Monte Carlo: Representation

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  $$\subseteq \mathsf{WSam}\, X \times \mathsf{List}\, \mathbb{I}$$
- $w_{-} : \sum_{t \in \mathsf{WSam}\, X} \mathrm{Paths}\, t \to \mathbb{R}_{+}$
  $w_{\mathsf{Return}\,(r,x)}([\,]) = r$
  $w_{\mathsf{Sample}\, t_{-}}(s :: r_s) = w_{t_s}(r_s)$

Trace Markov Chain Monte Carlo: Representation

Program traces

- $t \in \mathsf{WSam}\,X$: program structure representation
- $p : \mathsf{List}\,\mathbb{I}$ a trace in program t
- $$\sum_{t \in \mathsf{WSam}\,X} \mathrm{Paths}\,t := \left\{(t, p) \in \mathsf{WSam}\,X \times \mathsf{List}\,\mathbb{I} \middle| p \in t\right\}$$
  $$\subseteq \mathsf{WSam}\,X \times \mathsf{List}\,\mathbb{I}$$
- $w_{-} : \sum_{t \in \mathsf{WSam}\,X} \mathrm{Paths}\,t \to \mathbb{R}_{+}$
- $v_{-} : \sum_{t \in \mathsf{WSam}\,X} \mathrm{Paths}\,t \to X$
  $v_{\mathsf{Return}\,(r,x)}([\,]) = x$
  $v_{\mathsf{Sample}\,t_{-}}(s :: r_s) = v_{t_s}(r_s)$

Trace Markov Chain Monte Carlo: Representation

Program traces

- $t \in \mathsf{WSam}\, X$: program structure representation
- $p : \mathsf{List}\, \mathbb{I}$ a trace in program t
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- $w_{-} : \sum_{t \in \mathsf{WSam}\, X} \mathrm{Paths}\, t \to \mathbb{R}_{+}$
- $v_{-} : \sum_{t \in \mathsf{WSam}\, X} \mathrm{Paths}\, t \to X$

Tracing representation

$$\mathrm{Tr}\, \underline{T}\ X :=$$

$$\left\{ (t, a) \in \mathsf{WSam}\, X \times T(\mathsf{List}\, \mathbb{I}) \middle| \begin{array}{l} [\ \in t]\ m_{\underline{T}}(a)\text{-a.e., and} \\ m_{\mathsf{WSam}}(t) = \fint_{\mathsf{List}\, \mathbb{I}} \underline{\delta}_{v_t(p)}\, m_{\underline{T}}(a)(\mathrm{d}p) \end{array} \right\}$$

Trace Markov Chain Monte Carlo: Representation

Tracing representation

$$\mathrm{Tr}\,\underline{T}\ X :=$$

$$\left\{ (t,a) \in \mathsf{WSam}\,X \times T(\mathsf{List}\,\mathbb{I}) \;\middle|\; \begin{array}{l} [\,\in t]\ m_{\underline{T}}(a)\text{-a.e., and} \\ m_{\mathsf{WSam}}(t) = \oint_{\mathsf{List}\,\mathbb{I}} \underline{\delta}_{v_t(p)}\, m_{\underline{T}}(a)(\mathrm{d}p) \end{array} \right\}$$

$$\begin{array}{ll}
\multicolumn{2}{l}{\textbf{instance } \mathit{Inf} \implies \mathit{Inf}\ \ \mathit{Monad}\,(\mathrm{Tr}\,\underline{T})\ \textbf{where}} \\
\quad \textbf{return}\,x & = (\mathrm{return}_{\mathsf{WSam}}\,x, \mathrm{return}_{\underline{T}}[\ ]) \\
\quad (t,a) \gg\!= (f,g) & = (t \gg\!=_{\mathsf{WSam}} f, \underline{T}.\textbf{do}\,\{p \leftarrow a; \\
 & \qquad\qquad\qquad\qquad q \leftarrow g \circ v_t(p); \\
 & \qquad\qquad\qquad\qquad \textbf{return}(p +\!\!+ q))\}) \\
\quad m((,t),a) & = m_{\mathsf{WSam}}(t) = \oint_{\mathsf{List}\,\mathbb{I}} \underline{\delta}_{v_t(p)}\, m_{\underline{T}}(a)(\mathrm{d}p) \\
\quad \mathrm{sample} & = (\textbf{sample}_{\mathsf{WSam}}, \\
 & \quad \underline{T}.\textbf{do}\,\{r \leftarrow \mathrm{sample}; \textbf{return}[r]\}) \\
\quad \textbf{score}\,r & = (\textbf{sample}_{\mathsf{WSam}}, \\
 & \quad \underline{T}.\textbf{do}\,\{\textbf{score}\,r; \textbf{return}[\ ]\})
\end{array}$$

Markov Chain Monte Carlo: Transformation

Trace MCMC morphism

$$\eta^{\mathrm{Tr}\,T}_{\psi,\rho} : \mathrm{Tr}\,T\,X \to \mathrm{Tr}\,T\,X$$
$$\eta^{\mathrm{Tr}\,T}_{\psi,\rho}(t,a) := \big(t, \eta_{\psi_t,\rho_t}(a)\big)$$

Concrete proposal kernel and derivative

$$\psi_t : \mathsf{List}(\mathbb{I}) \to T(\mathsf{List}(\mathbb{I}))$$
$$\begin{aligned}\psi_t(p) := \underline{T}.\mathbf{do}\,\{&i \leftarrow \mathrm{U_D}^{\underline{T}}(|p|)\\ &q \leftarrow \mathrm{pri}^{T}(\mathrm{sub}(t, \mathrm{take}(i,p)))\\ &\mathbf{return}(\mathrm{take}(i,p) + q)\}\end{aligned}$$

$$\rho_t : \mathsf{List}(\mathbb{I}) \times \mathsf{List}(\mathbb{I}) \to \overline{\mathbb{R}}_+$$
$$\rho_t(p,q) := \frac{w_t(q)\cdot(|p|+1)}{w_t(p)\cdot(|q|+1)}$$

Contribution

Correctness of inference

- Modular validation of inference algorithms:
  Sequential Monte Carlo, Trace Markov Chain Monte Carlo
  By combining:
- Synthetic measure theory [Kock'12]: measure theory without measurable spaces
- Quasi-Borel spaces: a convenient category for higher-order measure theory [LICS'17]

Conclusion

Summary

- Bayesian inference: (continuous) sampling and conditioning
- Inference representation: monadic interface, sampling, conditioning, and meaning
- Plenty of opportunities for traditional programming language expertise

Further topics

- Sequential Monte Carlo (SMC)
- Combining SMC and MCMC into Move-Resample SMC
- Categorical structure of quasi-borel spaces