Algebraic Foundations for Effect-Dependent Optimisations

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39th Symposium on
Principles of Programming Languages
Philadelphia
January 26, 2012

Gifford-style types and effects

Effect systems

$$\ell_1 := 1;$$
$$\ell_2 := \mathbf{deref}(\ell_3)$$

Gifford-style types and effects

Effect systems

$$\vdash \ell_1 := 1; \quad \ell_2 := \mathbf{deref}(\ell_3) : () \,!\, \underbrace{\{\mathtt{lookup}, \mathtt{update}\}}_{\varepsilon}$$

$$\Gamma \vdash M : A \,!\, \varepsilon$$

Effect-dependent optimisations [Benton et al.]

Swap:

$$\begin{array}{c} \vdash M_i : () \,!\, \varepsilon_i, \\ \varepsilon_i \subseteq \{\mathtt{lookup}\} \end{array} \implies \begin{array}{c} M_1; M_2; N \\ = \\ M_2; M_1; N \end{array}$$

A language a paper

- N. Benton and A. Kennedy. *Monads, effects and transformations*, 1999.
- N. Benton, A. Kennedy, L. Beringer, M. Hofmann. *Reading, writing and relations*, 2006.
- N. Benton and P. Buchlovsky. *Semantics of an effect analysis for exceptions*, 2007.
- N. Benton, A. Kennedy, L. Beringer, M. Hofmann. *Relational semantics for effect-based program transformations with dynamic allocation*, 2007.
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Contribution

Craft

case by case treatment

$\Downarrow$

Science

general semantic account of Gifford-style effect type systems

$\Downarrow$

Engineering

- results: validate optimisations that occur in practice
- tools: to assist validation and instrumentation, e.g. optimisation tables
- methods: for overcoming difficulties, e.g. equational reasoning for modular validation

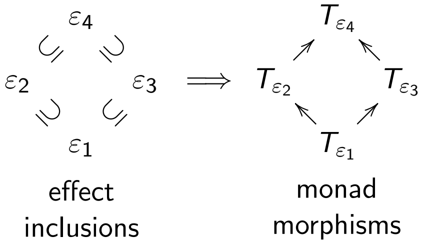
Marriage of effects and monads [Wadler and Thiemann]

Observation [Wadler]

Change notation:

$$\Gamma \vdash M : A \,!\, \varepsilon \implies \Gamma \vdash M : T_\varepsilon A$$

an indexed family $T_\varepsilon A$ of monadic types.



Suggested monads for global state

$$T_{\{\texttt{lookup},\texttt{update}\}}(A) = (2 \times A)^2$$

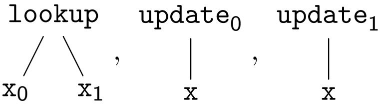
$$T_{\{\texttt{lookup}\}}(A) = A^2 \qquad T_{\{\texttt{update}\}}(A) = (1 + 2) \times A$$

$$T_{\emptyset}(A) = A$$

Algebraic theory of effects [Plotkin and Power]

An interface to effects:

Effect operations Σ e.g.: lookup : 2, update : 1 $\langle 2 \rangle$

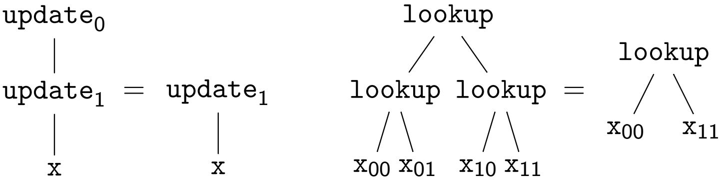


Algebraic theory of effects [Plotkin and Power]

An interface to effects:

Effect operations Σ e.g.: $\texttt{lookup} : 2$, $\texttt{update} : 1\ \langle 2 \rangle$

Effect equations E e.g.:



Each theory $\langle \Sigma, E \rangle$ **generates** a monad T (free model).

Algebraic view

Key observation

ε as an algebraic **signature**.

Global state

For $\Sigma := \{\texttt{lookup} : 2, \texttt{update} : 1 \langle 2 \rangle\}$,

$$\varepsilon = \emptyset, \{\texttt{lookup}\}, \{\texttt{update}\}, \{\texttt{lookup}, \texttt{update}\}$$

A **novel** banality.

Conservative restriction

Global state

$$E :=$$

$$\text{Theory}\left\{ \begin{array}{l} \mathtt{update}_b(\mathtt{update}_{b'}(\mathtt{x})) = \mathtt{update}_{b'}(\mathtt{x}), \\ \mathtt{lookup}(\mathtt{update}_0(\mathtt{x}), \mathtt{update}_1(\mathtt{x})) = \mathtt{x}, \\ \mathtt{update}_b(\mathtt{lookup}(\mathtt{x}_0, \mathtt{x}_1)) = \mathtt{update}_b(\mathtt{x}_b) \end{array} \right\}$$

Conservative restriction

Global state

$E_\varepsilon = \{s = t \in E \mid s,\ t \text{ are } \varepsilon\text{-terms}\}$

$E_{\{\mathtt{lookup},\mathtt{update}\}}$

$E_{\{\mathtt{lookup}\}} \subseteq E_{\{\mathtt{lookup},\mathtt{update}\}}$, $E_{\{\mathtt{update}\}} \subseteq E_{\{\mathtt{lookup},\mathtt{update}\}}$

$E_\emptyset \subseteq E_{\{\mathtt{lookup}\}}$, $E_\emptyset \subseteq E_{\{\mathtt{update}\}}$

$$E_{\{\mathtt{lookup},\mathtt{update}\}} = \text{Theory}\left\{\begin{array}{l} \mathtt{update}_b(\mathtt{update}_{b'}(\mathtt{x})) = \mathtt{update}_{b'}(\mathtt{x}), \\ \mathtt{lookup}(\mathtt{update}_0(\mathtt{x}), \mathtt{update}_1(\mathtt{x})) = \mathtt{x}, \\ \mathtt{update}_b(\mathtt{lookup}(\mathtt{x}_0, \mathtt{x}_1)) = \mathtt{update}_b(\mathtt{x}_b) \end{array}\right\}$$

Conservative restriction

Global state

$E_\varepsilon = \{s = t \in E \mid s,\ t \text{ are } \varepsilon\text{-terms}\}$

$$E_{\{\mathtt{lookup},\mathtt{update}\}} \supseteq E_{\{\mathtt{lookup}\}},\quad E_{\{\mathtt{lookup},\mathtt{update}\}} \supseteq E_{\{\mathtt{update}\}}$$

$$E_{\{\mathtt{lookup}\}} \supseteq E_\emptyset,\quad E_{\{\mathtt{update}\}} \supseteq E_\emptyset$$

$$E_{\{\mathtt{lookup}\}} =$$

$$\text{Theory}\left\{\begin{array}{l} \mathtt{lookup}(\mathtt{lookup}(\mathtt{x}_{00}, \mathtt{x}_{01}), \mathtt{lookup}(\mathtt{x}_{10}, \mathtt{x}_{11})) = \mathtt{lookup}(\mathtt{x}_{00}, \mathtt{x}_{11}), \\ \mathtt{lookup}(\mathtt{x}, \mathtt{x}) = \mathtt{x} \end{array}\right\}$$

Conservative restriction

Global state

$E_\varepsilon = \{ s = t \in E \mid s,\ t \text{ are } \varepsilon\text{-terms} \}$

$$\begin{array}{ccc} & E_{\{\texttt{lookup},\texttt{update}\}} & \\ \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\supseteq$} \\ E_{\{\texttt{lookup}\}} & & E_{\{\texttt{update}\}} \\ \rotatebox{-45}{$\supseteq$} & & \rotatebox{45}{$\subseteq$} \\ & E_\emptyset & \end{array}$$

$E_{\{\texttt{lookup}\}} =$

$$\mathrm{Theory}\left\{ \begin{array}{l} \texttt{lookup}(\texttt{lookup}(\texttt{x}_{00}, \texttt{x}_{01}), \texttt{lookup}(\texttt{x}_{10}, \texttt{x}_{11})) = \texttt{lookup}(\texttt{x}_{00}, \texttt{x}_{11}), \\ \texttt{lookup}(\texttt{x}, \texttt{x}) = \texttt{x} \end{array} \right\}$$

Reminder:

$E =$

$$\mathrm{Theory}\left\{ \texttt{update}_b(\texttt{update}_{b'}(\texttt{x})) = \texttt{update}_{b'}(\texttt{x}),\ \texttt{lookup}(\texttt{update}_0(\texttt{x}), \texttt{update}_1(\texttt{x})) = \texttt{x},\ \texttt{update}_b(\texttt{lookup}(\texttt{x}_0, \texttt{x}_1)) = \texttt{update}_b(\texttt{x}_b) \right\}$$

Conservative restriction

Global state

$E_\varepsilon = \{s = t \in E \,|\, s,\ t \text{ are } \varepsilon\text{-terms}\}$

$$E_{\{\texttt{lookup},\texttt{update}\}}$$

$$E_{\{\texttt{lookup}\}} \subseteq E_{\{\texttt{lookup},\texttt{update}\}} \supseteq E_{\{\texttt{update}\}}$$

$$E_{\{\texttt{lookup}\}} \supseteq E_\emptyset \subseteq E_{\{\texttt{update}\}}$$

$E_{\{\texttt{update}\}} =$

$$\text{Theory}\left\{ \texttt{update}_b(\texttt{update}_{b'}(\texttt{x})) = \texttt{update}_{b'}(\texttt{x}) \right\}$$

Conservative restriction

Global state

$E_\varepsilon = \{s = t \in E \,|\, s,\ t \text{ are } \varepsilon\text{-terms}\}$

$$E_{\{\texttt{lookup},\texttt{update}\}}$$

$$E_{\{\texttt{lookup}\}} \subseteq E_{\{\texttt{lookup},\texttt{update}\}} \qquad E_{\{\texttt{update}\}} \subseteq E_{\{\texttt{lookup},\texttt{update}\}}$$

$$E_\emptyset \subseteq E_{\{\texttt{lookup}\}} \qquad E_\emptyset \subseteq E_{\{\texttt{update}\}}$$

$$E_\emptyset = \mathrm{Theory}\,\emptyset$$

Derived monads

$$T_{\{\mathtt{lookup},\mathtt{update}\}}(A) = (2 \times A)^2$$

$$T_{\{\mathtt{lookup}\}}(A) = A^2 \qquad T_{\{\mathtt{update}\}}(A) = (1 + 2) \times A$$

$$T_{\emptyset}(A) = A$$

Optimisations

Structural properties

Valid for all T_ε

e.g.

- β, η rules
- sequencing

$$(M; N); P = M; (N; P)$$

Practically

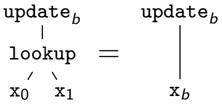
Bread and butter of optimisation, e.g.

- constant propagation
- common subexpression elimination
- loop unrolling

etc..

Local algebraic properties

Single equations in E_ε, e.g.:



become optimisations, e.g.:

$$\begin{array}{lcr} \ell := V; & & \mathtt{x} := V; \\ \mathtt{y} \leftarrow \mathtt{deref}(\mathtt{x}); & = & \\ N & & N[V/\mathtt{y}] \end{array}$$

note quantification over variables only (**local** property).

Global algebraic properties

Algebraic characterisation

For all $t(\mathtt{x}_1, \ldots, \mathtt{x}_n)$:

$$\begin{array}{ccc} & t & \\ \diagup & & \diagdown \\ \mathtt{x} & \cdots & \mathtt{x} \end{array} = \mathtt{x}$$

note quantification over **terms** too (**global** property).

Discard

$$M; \mathtt{return}\,() = \mathtt{return}\,()$$

Knowledge unification

Discard	B F	$\dfrac{\Gamma \vdash_{\varepsilon} M : \mathbf{F}_{\varepsilon} A \quad \Gamma \vdash_{\varepsilon'} N : \underline{B}}{(\mathbf{coerce}\, M)\ \mathbf{to}\ x : A.\, N = N}$	B F	$\dfrac{\Gamma \vdash_{\varepsilon} M : \mathbf{F}_{\varepsilon} A}{M\ \mathbf{to}\ x : A.\, \mathbf{return}_{\varepsilon} \star = \mathbf{return}_{\varepsilon} \star}$

	F $\mathcal{T}_{\varepsilon}$ affine: $\eta^{\varepsilon}_{\mathbb{1}} : \mathbb{1} \to \lvert F_{\varepsilon} \mathbb{1} \rvert$ has a continuous inverse	For all ε-terms t: $t(\mathbf{x}, \ldots, \mathbf{x}) = \mathbf{x}$

Knowledge unification

Figure 7. Abstract Optimisations

name	utilitarian form	pristine form	abstract side condition	algebraic equivalent	example basic theories
Discard	B F $\dfrac{\Gamma \vdash_\varepsilon M : \mathbf{F}_\varepsilon A \quad \Gamma \vdash_{\varepsilon'} N : \underline{B}}{(\mathbf{coerce}\,M)\ \mathbf{to}\ x : A.\, N = N}$	B F $\dfrac{\Gamma \vdash_\varepsilon M : \mathbf{F}_\varepsilon A}{M\ \mathbf{to}\ x : A.\, \mathbf{return}_\varepsilon \star = \mathbf{return}_\varepsilon \star}$	$\mathcal{T}_\varepsilon$ affine: F $\eta^\varepsilon_{\mathbb{1}} : \mathbb{1} \to \lvert F_\varepsilon \mathbb{1} \rvert$ has a continuous inverse	For all ε-terms t: $t(\mathbf{x}, \ldots, \mathbf{x}) = \mathbf{x}$	read-only state, convex, upper and lower semilattices
Copy	B F $\dfrac{\Gamma \vdash_\varepsilon M : \mathbf{F}_\varepsilon A \quad \Gamma, x : A, y : A \vdash_{\varepsilon'} N : \underline{B}}{\mathbf{coerce}\,M\ \mathbf{to}\ x : A.\ \mathbf{coerce}\,M\ \mathbf{to}\ y : A.\, N = \mathbf{coerce}\,M\ \mathbf{to}\ x : A.\, N[x/y]}$	B F $\dfrac{\Gamma \vdash_\varepsilon M : \mathbf{F}_\varepsilon A}{M\ \mathbf{to}\ x : A.\ M\ \mathbf{to}\ y : A.\, \mathbf{return}_\varepsilon(x, y) = M\ \mathbf{to}\ x : A.\, \mathbf{return}_\varepsilon(x, x)}$	$\mathcal{T}_\varepsilon$ relevant: F $\psi_\varepsilon \circ \delta = L^\varepsilon \delta$	For all ε-terms t: $t(t(\mathbf{x}_{11}, \ldots, \mathbf{x}_{1n}), \ldots, t(\mathbf{x}_{n1}, \ldots, \mathbf{x}_{nn})) = t(\mathbf{x}_{11}, \ldots, \mathbf{x}_{nn})$	exceptions, lifting, read-only state, write-only state
Weak Copy	$\dfrac{\Gamma \vdash_\varepsilon M : \mathbf{F}_\varepsilon A \quad \Gamma, y : A \vdash_{\varepsilon'} N : \underline{B}}{\mathbf{coerce}\,M\ \mathbf{to}\ x : A.\ \mathbf{coerce}\,M\ \mathbf{to}\ y : A.\, N = \mathbf{coerce}\,M\ \mathbf{to}\ y : A.\, N}$	$\dfrac{\Gamma \vdash_\varepsilon M : \mathbf{F}_\varepsilon A}{M\ \mathbf{to}\ x : A.\, M = M}$	$\mu^\varepsilon \circ L^\varepsilon \pi_1 \circ \mathrm{str}^\varepsilon \circ \delta = \mathrm{id}$	For all ε-terms t: $t(t(\mathbf{x}_1, \ldots, \mathbf{x}_n), \ldots, t(\mathbf{x}_1, \ldots, \mathbf{x}_n)) = t(\mathbf{x}_1, \ldots, \mathbf{x}_n)$	any affine or relevant theory: lifting, exceptions, read-only and write-only state, all three semilattice theories
Swap	B F $\dfrac{\Gamma \vdash_{\varepsilon_1} M_1 : \mathbf{F}_{\varepsilon_1} A_1 \quad \Gamma \vdash_{\varepsilon_2} M_2 : \mathbf{F}_{\varepsilon_2} A_2 \quad \Gamma, x_1 : A_1, x_2 : A_2 \vdash_{\varepsilon'} N}{\mathbf{coerce}\,M_1\ \mathbf{to}\ x_1 : A_1.\ \mathbf{coerce}\,M_2\ \mathbf{to}\ x_2 : A_2.\, N = \mathbf{coerce}\,M_2\ \mathbf{to}\ x_2 : A_2.\ \mathbf{coerce}\,M_1\ \mathbf{to}\ x_1 : A_1.\, N}$	B F $\dfrac{\Gamma \vdash_{\varepsilon_1} M_1 : \mathbf{F}_{\varepsilon_1} A_1 \quad \Gamma \vdash_{\varepsilon_2} M_2 : \mathbf{F}_{\varepsilon_2} A_2}{\mathbf{coerce}\,M_1\ \mathbf{to}\ x_1 : A_1.\ \mathbf{coerce}\,M_2\ \mathbf{to}\ x_2 : A_2.\, \mathbf{return}_\varepsilon\,(x_1, x_2) = \mathbf{coerce}\,M_2\ \mathbf{to}\ x_2 : A_2.\ \mathbf{coerce}\,M_1\ \mathbf{to}\ x_1 : A_1.\, \mathbf{return}_\varepsilon\,(x_1, x_2)}$	$\mathfrak{T}_{\varepsilon_1 \subseteq \varepsilon}, \mathfrak{T}_{\varepsilon_2 \subseteq \varepsilon}$ commute: F $\psi_\varepsilon \circ (m^{\varepsilon_1 \subseteq \varepsilon} \times m^{\varepsilon_2 \subseteq \varepsilon}) = \tilde{\psi}_\varepsilon \circ (m^{\varepsilon_1 \subseteq \varepsilon} \times m^{\varepsilon_2 \subseteq \varepsilon})$	$\mathfrak{T}_{\varepsilon_1 \subseteq \varepsilon}$ translations commute with $\mathfrak{T}_{\varepsilon_2 \subseteq \varepsilon}$ translations (see tensor equations)	$\mathcal{T}_1 \to \mathcal{T}_1 \otimes \mathcal{T}_2 \leftarrow \mathcal{T}_2$, e.g., distinct global memory cells
Weak Swap	$\dfrac{\Gamma \vdash_{\varepsilon_1} M_1 : \mathbf{F}_{\varepsilon_1} A_1 \quad \Gamma \vdash_{\varepsilon_2} M_2 : \mathbf{F}_{\varepsilon_2} A_2 \quad \Gamma, x_1 : A_1 \vdash_{\varepsilon'} N}{\text{(same as Swap)}}$	$\dfrac{\Gamma \vdash_{\varepsilon_1} M_1 : \mathbf{F}_{\varepsilon_1} A_1 \quad \Gamma \vdash_{\varepsilon_2} M_2 : \mathbf{F}_{\varepsilon_2} A_2}{\mathbf{coerce}\,M_1\ \mathbf{to}\ x_1 : A_1.\ \mathbf{coerce}\,M_2\ \mathbf{to}\ x_2 : A_2.\, \mathbf{return}_\varepsilon x_1 = \mathbf{coerce}\,M_2\ \mathbf{to}\ x_2 : A_2.\ \mathbf{coerce}\,M_1\ \mathbf{to}\ x_1 : A_1.\, \mathbf{return}_\varepsilon x_1}$	$\psi_\varepsilon \circ (m^{\varepsilon_1} \times m^{\varepsilon_2}) \circ (\mathrm{id} \times \eta^{\varepsilon_2}_{\mathbb{1}}) = \tilde{\psi}_\varepsilon \circ (m^{\varepsilon_1} \times m^{\varepsilon_2}) \circ (\mathrm{id} \times \eta^{\varepsilon_2}_{\mathbb{1}})$	For all ε-terms $t = \mathfrak{T}_1(t')$, $s = \mathfrak{T}_2(s')$: $t(s(\mathbf{x}_1, \ldots, \mathbf{x}_1), \ldots, s(\mathbf{x}_n, \ldots, \mathbf{x}_n)) = s(t(\mathbf{x}_1, \ldots, \mathbf{x}_n), \ldots, t(\mathbf{x}_1, \ldots, \mathbf{x}_n))$	when $\mathcal{T}_{\varepsilon_2}$ is affine, e.g.: read-only state and convex, upper and lower semilattices.
Isolated Swap	$\dfrac{\Gamma \vdash_{\varepsilon_1} M_1 : \mathbf{F}_{\varepsilon_1} A_1 \quad \Gamma \vdash_{\varepsilon_2} M_2 : \mathbf{F}_{\varepsilon_2} A_2 \quad \Gamma \vdash_{\varepsilon'} N}{\text{(same as Swap)}}$	$\dfrac{\Gamma \vdash_{\varepsilon_1} M_1 : \mathbf{F}_{\varepsilon_1} A_1 \quad \Gamma \vdash_{\varepsilon_2} M_2 : \mathbf{F}_{\varepsilon_2} A_2}{\mathbf{coerce}\,M_1\ \mathbf{to}\ x_1 : A_1.\ \mathbf{coerce}\,M_2\ \mathbf{to}\ x_2 : A_2.\, \mathbf{return}_\varepsilon \star = \mathbf{coerce}\,M_2\ \mathbf{to}\ x_2 : A_2.\ \mathbf{coerce}\,M_1\ \mathbf{to}\ x_1 : A_1.\, \mathbf{return}_\varepsilon \star}$	$\psi_\varepsilon \circ (m^{\varepsilon_1} \times m^{\varepsilon_2}) \circ (\eta^{\varepsilon_1}_{\mathbb{1}} \times \eta^{\varepsilon_2}_{\mathbb{1}}) = \tilde{\psi}_\varepsilon \circ (m^{\varepsilon_1} \times m^{\varepsilon_2}) \circ (\eta^{\varepsilon_1}_{\mathbb{1}} \times \eta^{\varepsilon_2}_{\mathbb{1}})$	For all ε-terms $t = \mathfrak{T}_1(t')$, $s = \mathfrak{T}_2(s')$: $t(s(\mathbf{x}, \ldots, \mathbf{x}), \ldots, s(\mathbf{x}, \ldots, \mathbf{x})) = s(t(\mathbf{x}, \ldots, \mathbf{x}), \ldots, t(\mathbf{x}, \ldots, \mathbf{x}))$	when $\mathcal{T}\varepsilon_1$ is affine: read-only state and convex, upper and lower semilattices.
Unique	B $\dfrac{\Gamma \vdash_\varepsilon M_i : \mathbf{F}_\varepsilon \mathbf{0}, i = 1, 2}{M_1 = M_2}$	(same as utilitarian form)	$F_\varepsilon \mathbb{0} = \mathbb{0}, \mathbb{1}$	$\mathcal{T}_\varepsilon$ equates all ε-constants	all three state theories, all three semilattice theories, a single unparameterised exception, lifting
Pure Hoist	B F $\dfrac{\Gamma \vdash_\varepsilon M : \mathbf{F}_\varepsilon A \quad \Gamma, x : A \vdash_{\varepsilon'} N : \underline{B}}{\mathbf{return}_\varepsilon\ \mathbf{thunk}\ (\mathbf{coerce}\,M\ \mathbf{to}\ x : A.\, N) = M\ \mathbf{to}\ x : A.\, \mathbf{return}_\varepsilon\ \mathbf{thunk}\ N}$	B F $\dfrac{\Gamma \vdash_\varepsilon M : \mathbf{F}_\varepsilon A}{\mathbf{return}_\varepsilon\ \mathbf{thunk}\ M = M\ \mathbf{to}\ x : A.\, \mathbf{return}_\varepsilon\ \mathbf{thunk}\ \mathbf{return}_\varepsilon x}$	F $L^\varepsilon \eta^\varepsilon_W = \eta^\varepsilon_{\lvert F_\varepsilon W \rvert}$	all ε-terms are equal to variables in $\mathcal{T}_\varepsilon$	the empty theory, inconsistent theories
Hoist	B F $\dfrac{\Gamma \vdash_\varepsilon M : \mathbf{F}_\varepsilon A \quad \Gamma, x : A \vdash_{\varepsilon'} N : \underline{B}}{M\ \mathbf{to}\ x : A.\ \mathbf{return}_\varepsilon\ \mathbf{thunk}\ (\mathbf{coerce}\,M\ \mathbf{to}\ x : A.\, N) = M\ \mathbf{to}\ x : A.\, \mathbf{return}_\varepsilon\ \mathbf{thunk}\ N}$	B F $\dfrac{\Gamma \vdash_\varepsilon M : \mathbf{F}_\varepsilon A}{M\ \mathbf{to}\ x : A.\ \mathbf{thunk}\ \mathbf{return}_\varepsilon\,(x, \mathbf{thunk}\ M) = M\ \mathbf{to}\ x : A.\ \mathbf{thunk}\ \mathbf{return}_\varepsilon\,(x, \mathbf{thunk}\ \mathbf{return}_\varepsilon x)}$	F $L^\varepsilon \langle \eta^\varepsilon, \mathrm{id} \rangle = \mathrm{str}^\varepsilon \circ \delta$	all ε-terms are either a variable or independent of their variables via $\mathcal{T}_\varepsilon$	all theories containing only constants: lifting and exceptions

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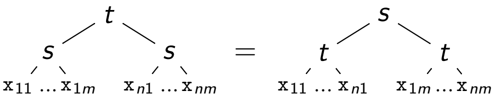
Global algebraic properties

Algebraic characterisation

For all ε_1-term $t(\mathtt{x}_1, \ldots, \mathtt{x}_n)$, and ε_2-term $s(\mathtt{x}_1, \ldots, \mathtt{x}_m)$:



Swap

$$\mathtt{x} \leftarrow M_1; \mathtt{y} \leftarrow M_2; \mathtt{return}\ \langle \mathtt{x}, \mathtt{y} \rangle$$

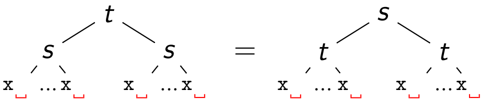
$$=$$

$$\mathtt{y} \leftarrow M_2; \mathtt{x} \leftarrow M_1; \mathtt{return}\ \langle \mathtt{x}, \mathtt{y} \rangle$$

Global algebraic properties

Algebraic characterisation

For all ε_1-term $t(\mathtt{x}_1, \ldots, \mathtt{x}_n)$, and ε_2-term $s(\mathtt{x}_1, \ldots, \mathtt{x}_m)$:



Isolated swap

$$M; N = N; M$$

Applicable for more effects.

Additional contributions

Details in the paper, and:

- An extended **example**:

$$\text{Exceptions} + (\text{Read Only} \otimes \text{Write Only} \otimes \text{Read-Write} \otimes (\text{Rollback Exceptions} + \text{Input} + \text{Output} + (\text{Non-determinism} \otimes \text{Lifting})))$$

  ($2^9 = 512$ effect sets).
- **Modular validation** of optimisations.
- Guaranteeing optimisation **soundness**.
- Optimisation tables.

Caveats

- No effect inference.
- Not a rich logic (equational only).
- Only algebraic effects.
- Did not cover all optimisations.

Summary

- N. Benton and A. Kennedy. *Monads, effects and transformations*, 1999.
- N. Benton, A. Kennedy, L. Beringer, M. Hofmann. *Reading, writing and relations*, 2006.
- N. Benton and P. Buchlovsky. *Semantics of an effect analysis for exceptions*, 2007.
- N. Benton, A. Kennedy, L. Beringer, M. Hofmann. *Relational semantics for effect-based program transformations with dynamic allocation*, 2007.
- N. Benton, A. Kennedy, L. Beringer, M. Hofmann. *Relational semantics for effect-based program transformations: higher-order store*, 2009.
- J. Thamsborg, L. Birkedal. *A kripke logical relation for effect-based program transformations*, 2011.

Summary

- **Category theory** was crucial to this formulation.
- The categorical characterisations connected to Führmann, Jacobs, Kock and Wraith.

Contribution

Craft

case by case treatment

$\Downarrow$

Science

general semantic account of Gifford-style effect type systems

$\Downarrow$

Engineering

- results: validate optimisations that occur in practice
- tools: to assist validation and instrumentation, e.g. optimisation tables
- methods: for overcoming difficulties, e.g. equational reasoning for modular validation

Appendices

- High-level view
- IR syntax
  - Signature
  - Types and terms
  - Type system
- IR semantics
- Optimisation soundness
- Atkey
- Further work

Appendix I: Bird's Eye

monadic source language

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multi-monadic intermediate language

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semantics

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optimisations (logic)

IR syntax

Signature

$\Sigma = \{op : a \langle p \rangle\}$ parametrises the language.

Global state

$$\begin{aligned} \text{State:}\ & \texttt{lookup} : 2\ (\texttt{lookup} : 2\,\langle 1 \rangle), \quad \texttt{update} : 1\,\langle 2 \rangle \\ \text{Exceptions:}\ & \texttt{DivideByZero} : 0 \\ \text{Input:}\ & \texttt{input} : 128,\ \texttt{output} : 1\,\langle 128 \rangle \end{aligned}$$

Already $\mathbf{2^5} = \mathbf{32}$ different languages!

IR syntax

Types and terms

$$
\begin{array}{rcll}
A, B, \ldots & ::= & \mathbf{n} \mid A \to B & \mid T_\varepsilon A \\
M, N, \ldots & ::= & x \mid i \mid \lambda x.M & \mid M\,N \\
 & & \mid \mathtt{return}_\varepsilon\, M & \mid x \leftarrow M; N \\
 & & \mid \mathtt{op}_M N & \mid M
\end{array}
$$

where $\varepsilon, \varepsilon' \subseteq \Sigma$

IR syntax

Type system

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M} \qquad \frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\,N : B}$$

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathtt{return}_{\varepsilon}\, M : T_{\varepsilon} A} \qquad \frac{\Gamma \vdash M : T_{\varepsilon} A \qquad \Gamma, x : A \vdash N : T_{\varepsilon} B}{\Gamma \vdash x \leftarrow M; N : T_{\varepsilon} B}$$

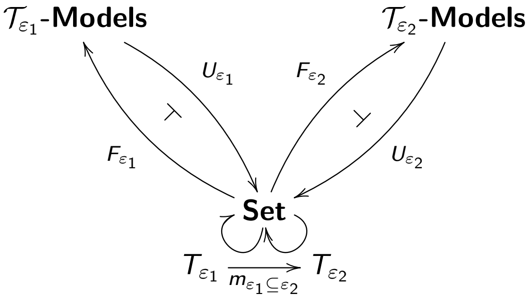
$$\frac{\Gamma \vdash M : \mathbf{p} \qquad \Gamma \vdash N : \mathbf{a} \to T_{\varepsilon} B}{\Gamma \vdash \mathtt{op}_M N : T_{\varepsilon} B}\ \mathtt{op} : a \langle p \rangle, \mathtt{op} \in \varepsilon$$

Semantics

Models

A functorial family of theories: $\mathcal{T}_\varepsilon = \langle \varepsilon, E_\varepsilon \rangle$
with $E_{\varepsilon_1} \subseteq E_{\varepsilon_2}$ whenever $\varepsilon_1 \subseteq \varepsilon_2$.

Derived monads



Effect-dependent optimisation

Source: $\mathtt{x} \leftarrow M; \mathtt{return}\, 0 \quad : T\mathbf{1}$

Effect-dependent optimisation

Source: $\mathtt{x} \leftarrow M; \mathtt{return}\, 0 \quad = \quad \mathtt{return}\, 0 \quad : T\mathbf{1}$

$$\mathtt{x} \leftarrow M;$$

IR: $\mathtt{return}_{\{\mathtt{lookup}\}}\, 0 \quad = \quad \mathtt{return}_{\{\mathtt{lookup}\}}\, 0 \quad : T_{\{\mathtt{lookup}\}}\mathbf{1}$

crucial step holds $\forall N : T_{\{\mathtt{lookup}\}}A$, not $\forall N : TA$

Effect-dependent optimisation

Source: $\mathtt{x} \leftarrow M; \mathtt{return}\, 0 \quad = \quad \mathtt{return}\, 0 \quad : T\mathbf{1}$

IR:

$$\begin{aligned} &\mathtt{x} \leftarrow M; \\ &\mathtt{return}_{\{\mathtt{lookup}\}}\, 0 \quad = \quad \mathtt{return}_{\{\mathtt{lookup}\}}\, 0 \quad : T_{\{\mathtt{lookup}\}}\mathbf{1} \\ &\qquad\qquad\qquad\qquad\qquad\; \mathtt{return}_{\emptyset}\, 0 \quad : T_{\emptyset}\mathbf{1} \end{aligned}$$

Formalising soundness

Erasure

Erase : IR terms $\to$ source terms

$\mathrm{Erase}(M)$: remove ε's and coercions from M

$$(\mathtt{x} \leftarrow M; \mathtt{return}_{\emptyset}\, 0) \quad \stackrel{\mathrm{Erase}}{\longmapsto} \quad \mathtt{x} \leftarrow \mathrm{Erase}(M); \mathtt{return}\, 0$$

Validity

$\mathcal{M}$ a model (source or IR):

$$\mathcal{M} \models M = N \stackrel{def}{\iff} [\![M]\!] = [\![N]\!] \text{ in } \mathcal{M}$$

Formal soundness

Soundness

For a **source** model $\mathcal{T}$ and IRs $\vdash M, N : T_\varepsilon \mathbf{n}$,
suffices to find an IR model $\mathcal{T}^\sharp$ such that:

$$\mathcal{T}^\sharp \models M = N \implies \mathcal{T} \models \mathrm{Erase}(M) = \mathrm{Erase}(N)$$

Source:	$\mathit{Erase}(M)$		$\mathit{Erase}(N)$	$: T\mathbf{n}$
IR:	M	$= M' = M'' = \ldots = M''' =$	N	$: T_\varepsilon \mathbf{n}$

Constructing IR Models

Conservative Restriction Model

Given $\mathcal{T} = \langle \Sigma, E \rangle$, define the IR model $\mathcal{T}^{\mathrm{Cns}}$ by:

$$E|_{\varepsilon} := E \cap (\varepsilon\text{-terms} \times \varepsilon\text{-terms})$$

i.e., all derivable E equations between ε-terms.

Theorem

For all $\vdash M, N : T_{\varepsilon}\mathbf{n}$:

$$\mathcal{T}^{\mathrm{Cns}} \models M = N \iff \mathcal{T} \models \mathrm{Erase}(M) = \mathrm{Erase}(N)$$

Modularity theorem

Idea

Restrictions of $\mathcal{T} = \mathcal{T}^1 \bigcirc \mathcal{T}^2$ in terms of component restrictions.

Theorem

For consistent theories:

$$(\mathcal{T}^1 + \mathcal{T}^2)\big|_{\varepsilon_1 + \varepsilon_2} = \mathcal{T}^1\big|_{\varepsilon_1} + \mathcal{T}^2\big|_{\varepsilon_2}$$

Axiomatic restriction

Axiomatic Restriction Model

Given $\mathcal{T} = \langle \Sigma, \mathrm{TheoryAx} \rangle$, define the IR model $\mathcal{T}^{\mathrm{Ax}}$ by:

$$\mathrm{Theory}|_{\varepsilon} \mathrm{Ax} \coloneqq \mathrm{Theory}(\mathrm{Ax} \cap (\varepsilon\text{-terms} \times \varepsilon\text{-terms}))$$

By fiat,

$$\begin{aligned} \mathrm{Theory}|_{\varepsilon_1+\varepsilon_2} (\mathrm{Ax}^1 + \mathrm{Ax}^2) &= \mathrm{Theory}|_{\varepsilon_1} \mathrm{Ax}^1 + \mathrm{Theory}|_{\varepsilon_2} \\ \mathrm{Theory}|_{\varepsilon_1+\varepsilon_2} ((\mathrm{Ax}^1 + \mathrm{Ax}^2) \cup E_{\Sigma_1 \otimes \Sigma_2}) &= \mathrm{Theory}|_{\varepsilon_1} \mathrm{Ax}^1 \otimes \mathrm{Theory}|_{\varepsilon_2} \end{aligned}$$

Theorem

For all $\vdash M, N : T_{\varepsilon}\mathbf{n}$*:*

$$\mathcal{T}^{\mathrm{Ax}} \models M = N \implies \mathcal{T} \models \mathrm{Erase}(M) = \mathrm{Erase}(N)$$

(contd.) Discard: $\mathtt{x} \leftarrow M; \mathtt{return}_\varepsilon\, 0 = \mathtt{return}_\varepsilon\, 0$

Discard: Pristine Form

$$\frac{\Gamma \vdash M : T_\varepsilon A}{\mathtt{x} \leftarrow M; \mathtt{return}_\varepsilon\, 0 = \mathtt{return}_\varepsilon\, 0}$$

(cont.)

Categorical Characterisation

$$T_\varepsilon 1 \cong 1$$

Due to Kock, Jacobs, Führmann

Further work

- Effect reconstruction
- Handlers
- Automation
- More effects
- Locality
- Concurrency
- DSL reasoning.
- Richer program logics (Hoare, modal, etc.).

Isolated swap applicability

For example, if $\varepsilon_1 = \{\texttt{input}\}, \varepsilon_2 = \{\texttt{lookup}, \texttt{update}\}$.

Atkey

Precise relationship of semantics is further work.
Similarities:

- Soundness of optimisations.
- Validation of the Benton et. al global state optimisations.
- Constructing a semantics out of an equational theory.

Differences:

- Our work included a general treatment of optimisations.
- Our work is tightly coupled to the algebraic semantics.
- Out work treats modular combinations of optimisations.

Perhaps our work can be generalised to the parametrised setting.