A domain theory for quasi-Borel spaces

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Probabilistic programming

$$[\![-]\!] : \text{programs} \to \text{distributions}$$

- Continuous types: $\mathbb{R}$, $[0, \infty]$
- Probabilistic effects:

normally distributed sample

$$\textbf{sample} : \mathbb{R}$$

scale distribution by r

$$\frac{r : [0, \infty]}{\textbf{score}(r) : \mathbf{1}}$$

- Commutativity/Fubini

evaluation order doesn't matter: s-finite distributions [Staton'17]

- Traditional programming:
  - Higher-order functions
  - Inductive types and bounded iteration
  - Type and term recursion

quasi-Borel spaces [Heunen et al.'17]

modular implementation of Bayesian inference algorithms [Ścibior et al.'18]

domain theory [this work]

Iso-recursive types: FPC

[Fiore-Plotkin'94]

type variable contexts

$$\Delta = \{\alpha_1, \ldots, \alpha_n\} \qquad \frac{\Delta, \alpha \vdash_{\mathrm{k}} \tau : \mathsf{type}}{\Delta \vdash_{\mathrm{k}} \mu\alpha.\tau : \mathsf{type}}$$

type recursion

$$\frac{\Gamma \vdash t : \sigma[\alpha \mapsto \tau]}{\Gamma \vdash \tau.\mathbf{roll}\,(t) : \tau}(\tau = \mu\alpha.\sigma)$$

$$\frac{\Gamma \vdash t : \mu\alpha.\sigma \qquad \Gamma, x : \sigma[\alpha \mapsto \mu\alpha.\sigma] \vdash s : \tau}{\Gamma \vdash \mathbf{match}\, t \,\mathbf{with}\, \mathbf{roll}\, x \Rightarrow s : \tau}$$

locally continuous functor

$\omega\mathbf{Cpo}$-enriched category

$$[\![\Delta \vdash_{\mathrm{k}} \tau : \mathsf{type}]\!] : (\mathcal{C}^{\mathrm{op}})^n \times \mathcal{C}^n \to \mathcal{C}$$

Recursive types denote minimal invariants $[\![\Delta \vdash_{\mathrm{k}} \mu\alpha.\tau : \mathsf{type}]\!]$

[Pitts'96]

Challenge

- probabilistic powerdomain
- commutativity/Fubini
- domain theory
- higher-order functions

continuous domains [Jones-Plotkin'89]

open problem [Jung-Tix'98]

traditional approach:

domain $\mapsto$ open subsets $\mapsto$ Borel subsets $\mapsto$ distributions

our approach:

$$(\text{domain}, \text{quasi-Borel space}) \mapsto \text{distributions}$$

separate
but compatible

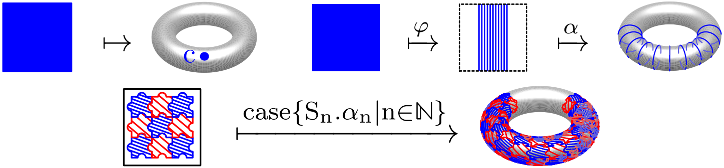
quasi-Borel pre-domains

ω-qbs:

set · subset of functions $\mathbb{R} \to \underline{X}$ · partial order on $\underline{X}$

$$X = (\underline{X}, M_X, \leq_X)$$

quasi-Borel space · ω-cpo



$\bigvee_n x_n$

$\vdots$

x_1

x_0

s.t.:

pointwise ω-chain · pointwise lub

$$(\alpha_n) \in M_X^{\omega} \implies \bigvee_n \alpha_n \in M_X$$

Morphisms $f : X \to Y$: Scott continuous qbs maps

monotone and
$f \bigvee_n x_n = \bigvee_n f\, x_n$

$\forall \alpha \in M_X.$
$f \circ \alpha \in M_Y$

Characterising $\omega\mathbf{Qbs}$

- qbs: $\mathbf{Sbs} \mapsto [\mathbf{Sbs}^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{cpp}} \mapsto$ SepSh

  Thm: $\mathbf{Qbs} \simeq$ SepSh

  countable product preserving

  canonical map $F\mathbb{R} \to (F\mathbb{1})^{\mathbb{R}}$ injective

  ω-qbs: $\mathbf{Sbs} \mapsto [\mathbf{Sbs}^{\mathrm{op}}, \omega\mathbf{Cpo}]_{\mathrm{cpp}} \mapsto \omega$SepSh

  Thm: $\omega\mathbf{Qbs} \simeq \omega$SepSh

  canonical map $F\mathbb{R} \to (F\mathbb{1})^{\mathbb{R}}$ full mono (order reflecting)

- ω-cpos internal to $\mathbf{Qbs}$ as a quasi-topos

  strong monos: $X \overset{f}{\rightarrowtail} Y$ $(f\circ)^{-1}[M_Y] = M_X$

- algebras for an essentially algebraic theory:

$$\boldsymbol{\omega}\mathbf{cpo} + \mathbf{qbs} + \text{compatibility axiom}$$

Axiomatic domain theory

[Fiore-Plotkin'94, Fiore'96]

strong mono

Borel-open map $m : X \rightarrowtail Y$:

qbses

$$\forall \beta \in M_Y. \qquad \beta^{-1}[m[X]] \in \mathcal{B}(\mathbb{R})$$

Borel-Scott open maps form a domain structure on $\omega\mathbf{Qbs}$

$\Longrightarrow$ model axiomatic domain theory

$\Longrightarrow$ solve recursive domain equations

A probabilistic powerdomain

$$\alpha \quad \mapsto \quad \lambda f : \underset{[0,\infty]}{\overset{X}{\downarrow_\omega}} . \int_{\alpha^{-1}[X]} f \circ \alpha(x) \lambda(\mathrm{d}x)$$

Lebesgue measure

$\alpha^{-1}[X] \xrightarrow{f\circ\alpha} [0,\infty]$ Borel

$$(X_\perp)^{\mathbb{R}} \longrightarrow \mathbb{L}^{\mathbb{L}^X}$$

$$(X_\perp)^{\mathbb{R}} \twoheadrightarrow MX \rightarrowtail \mathbb{L}^{\mathbb{L}^X}$$

Borel maps and natural order

continuation monad

where:

$$X \xrightarrow{f} Y \qquad X \twoheadrightarrow \left(\mathrm{Cl}_\omega f[X], \mathrm{Cl}_\omega^{Y^{\mathbb{R}}} f \circ [M_X], \leq_Y\right) \rightarrowtail Y$$

Fact: densely strong epis closed under: product, lifting, $(-)^{\mathbb{R}}$

$\implies$ monad-structure factorisation [McDermott-Kammar'18]

Summary

- $\omega\mathbf{Qbs}$: separate, compatible domain and qbs structures
- Cartesian-closed, locally presentable
- Borel-Scott open maps
  model axiomatic domain theory
- Commutative probabilistic powerdomain
  models synthetic measure theory
- Adequate semantics for Probabilistic FPC